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On Gauge Couplings in String Theory[⋆]

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ABSTRACT

We investigate the field dependence of the gauge couplings of $N = 1$ string vacua from the point of view of the low energy effective quantum field theory. We find that field-theoretical considerations severely constrain the form of the string loop corrections; in particular, the dilaton dependence of the gauge couplings is completely universal at the one-loop level. The moduli dependence of the string threshold corrections is also constrained, and we illustrate the power of such constraints with a detailed discussion of the orbifold vacua and the (2, 2) (Calabi-Yau) vacua of the heterotic string.

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1. Introduction and Summary

A unified fundamental theory of all known forces has been one of the prime goals of theoretical high-energy physics. Among the presently known candidates for such a unified theory, string theories appear to be free of the mathematical inconsistencies at short distances that plague the fundamentally-local quantum field theories. This mild ultraviolet behavior results from an infinite tower of superheavy particles in the string spectrum; nevertheless, the long-distance limit of a string theory can be described by an effective quantum field theory (EQFT) containing only a finite number of local fields. Of particular importance is the heterotic superstring[1] whose massless spectrum can comfortably include the $SU(3) \times SU(2) \times U(1)$ gauge fields of the Standard Model as well as families of chiral fermions with quantum numbers of the quarks and leptons.

Unlike conventional "Grand Unified Theories," string theory does not combine all the gauge forces into a single simple group; instead, at energies just below the string scale, the gauge group has a product structure $G = \prod_a G_a$. Nevertheless, at the tree level of string theory, all the simple factors G_a have related gauge couplings,[1,2]

$$g_a^{-2} \;=\; k_a \; g_{\rm string}^{-2}\,, \tag{1.1}$$

where $g_{\rm string}$ is the universal string coupling parameter and k_a denotes the normalization of the gauge group generators (in string-theoretical terms, k_a is the level of the Kač-Moody current algebra giving rise to G_a).

The universality of the tree-level gauge couplings (1.1) is spoiled at the loop level by the low-energy renormalization and by finite threshold corrections due to loops of charged superheavy particles that decouple from the low-energy EQFT.[3,4] At the one-loop level of the string theory (and also of the EQFT), the running effective gauge couplings are given by

$$g_a^{-2}(p) \;=\; k_a g_{\rm string}^{-2} \;+\; \frac{\mathbf{b}_a}{16\pi^2}\,\log\frac{M_{\rm string}^2}{p^2} + \frac{\Delta_a}{16\pi^2}\,, \tag{1.2}$$

where p is the momentum scale at which the effective couplings are measured (which is assumed to be much less than the mass of any superheavy string mode) and $\mathbf{b}_a/16\pi^2$ is the coefficient of the one-loop β-function of the low-energy EQFT. Similar to the threshold corrections in ordinary GUTs,[3] the one-loop string-threshold corrections Δ_a can be computed in terms of charges and masses of the superheavy string modes.[4]

The physical interest of studying the string-threshold corrections is twofold: First of all, as in any unified theory, Δ_a are part of the high-energy boundary conditions for the renormalization group equations for the gauge couplings of the Standard Model and thus affect their low-energy values; indeed, current electroweak measurements at LEP and SLC are precise enough to be sensitive to such threshold corrections.[5] Thus, in string-based models without additional, intermediate-scale thresholds in the observed sector, precision electroweak measurements impose stringent phenomenological constraints on the physics at the string scale. Note that for the string unification, the nominal unification scale (denoted by $M_{\rm string}$ in eq. (1.2)) is not a free parameter of the theory (like $M_{\rm GUT}$ in conventional GUTs) but a computable quantity; at the one-loop level of accuracy, $M_{\rm string} \approx g_{\rm string} \times 5 \cdot 10^{17}$ GeV.[4] Therefore, in string theory, the threshold corrections Δ_a have much stronger phenomenological impact than in GUTs and deserve serious investigation.[6–8]

The second, and for the present investigation more important aspect of the threshold corrections results from the extreme sensitivity of various low-energy non-perturbative effects to the ultraviolet values of the gauge couplings. A major problem of the string unification is that to all orders in perturbation theory, supersymmetric ground states of the heterotic string are not isolated from each other but come in continuous families of exactly degenerate vacua parametrized by the vacuum expectation values (VEVs) of gauge-neutral scalar fields Φ^i usually called moduli. Generally, all couplings of the low-energy EQFT depend on the moduli VEVs and hence remain undetermined until some non-perturbative effects induce a non-trivial effective potential for the moduli fields and lift the exact degeneracy of the perturbation theory. This effective potential is also essential for the spontaneous breakdown of spacetime supersymmetry at or just above the weak scale.[9,10]

It is of course possible that the non-perturbative effects giving rise to this effective potential are of an inherently stringy nature and thus are beyond our present knowledge.[11] However, there are good reasons to assume that the leading non-perturbative effects are due to infrared-strong interactions in a "hidden" sector of the low-energy EQFT.[12] The energy scale at which such interactions become strong — and thus the overall magnitude of all the field-theoretical non-perturbative effects — is controlled by asymptotically free gauge interactions, and hence the shape of the resulting effective potential for the moduli is extremely sensitive to the field dependence of the relevant gauge couplings.[9,10,13]

Generally, moduli dependence of the gauge couplings of a string-based EQFT can be studied in two very different ways. One approach is to calculate the gauge couplings directly from the string-theoretical amplitudes involving the gauge fields and then analyze their moduli dependence. At the tree level, this approach yields eq. (1.1) where $g_{\rm string}^{-2}$ depends solely on the dilaton — a modulus common to all vacuum families of the heterotic string. For the $N=1$ supersymmetric vacua, this dependence can be summarized as[14]

$$\left[f_a \equiv \frac{1}{g_a^2} - \frac{i\theta_a}{8\pi^2}\right]^{\rm tree} = k_a S \tag{1.3}$$

where S is (the bosonic part of) the dilaton/axion chiral superfield. At the one-loop level, the gauge couplings are given by eqs. (1.2), in which the threshold corrections Δ_a do not depend on the dilaton but generally do depend on all the other moduli M^i of the vacuum family. If the masses of the superheavy string modes are known as analytic functions of the moduli, then the moduli dependence of the Δ_a can also be evaluated in analytic form. Following this approach,

L. Dixon and the present authors[15] have calculated the Δ_a of factorizable $(0,2)$ orbifolds as explicit functions of the untwisted moduli; the same method was subsequently extended to other classes of string vacua in refs. 16–19.

The other approach to the moduli dependence of the gauge couplings is based on constraints due to local supersymmetry of the low-energy EQFT. In the Wilsonian action of the EQFT, a gauge coupling appears in a chiral superspace integral $\int d^4x d^2\Theta \mathcal{E}\, f_a(\Phi)\, \mathrm{tr}_a(W^\alpha W_\alpha)$ and hence has to be a harmonic function, *i.e.*, the real part of a holomorphic function $f_a(\Phi)$ of the complex moduli fields.[20] The chirality of this action for the gauge superfields leads to a powerful no-renormalization theorem: There are no perturbative corrections to the f_a beyond the one-loop level.[21,22,16] On the other hand, the Wilsonian gauge couplings of an EQFT do not account for the low-energy loops of the light fields and hence do not immediately connect to physical quantities such as scattering amplitudes.⋆ Instead, one may define more physical, momentum-dependent (running) *effective* gauge couplings, which are free of these problems, although they have complications of their own: The effective gauge couplings renormalize at all orders of the perturbation theory and their moduli dependence is non-harmonic.[15,24–30,13] However, this non-harmonicity is a purely low-energy effect and can be calculated from the low-energy EQFT without any knowledge of the superheavy particles; it is the harmonic terms in the moduli-dependent effective gauge couplings that are sensitive to the physics at the high-energy threshold. Such terms can always be interpreted as threshold corrections to the Wilsonian gauge couplings $f_a(\Phi)$, and because of the no-renormalization theorem, they can arise at the one-loop level of the perturbation theory or non-perturbatively, but not at any multi-loop

⋆ From the renormalization theory's point of view, the Wilsonian couplings are couplings of the EQFT from which the high-energy degrees of freedom are integrated out but the low-energy quantum operators are left as they are. On the other hand, the generating functional of the 1PI Feynman graphs defines a non-local effective classical action that summarizes all the quantum effects, both high-energy and low-energy. The distinction between the Wilsonian action and the effective classical action and between the corresponding couplings is described in detail in refs. 21,13,23.

level.

The effective gauge couplings are physical and hence invariant under any exact symmetry of the low-energy EQFT. However, in order to cancel the potential anomalies arising from chiral rotation and rescaling of the charged fermions, the Wilsonian gauge couplings may be subject to non-trivial transformation laws, which are determined at the one-loop level of the EQFT in terms of its tree-level couplings.[25–27,13] These anomalous transformation laws act as extremely powerful constraints on the holomorphic functions $f_a(\Phi)$; indeed, if the moduli space of the EQFT modded out by all the discrete symmetries were a compact non-singular manifold, the $f_a(\Phi)$ would be completely determined (up to constant terms) by their symmetry transformations alone.[13] More generally, the functional form of the $f_a(\Phi)$ is determined by their transformation properties and their asymptotic behaviors at the singular points of the moduli space and along its non-compact directions (*i.e.*, the large radius limit of a Calabi-Yau manifold).

The purpose of this article is to interrelate the string-theoretical and the field-theoretical approaches, to establish their mutual consistency and to demonstrate the power of the field-theoretical constraints in the context of string theory. In the following section (2), we discuss generic properties of four-dimensional, $N=1$ supersymmetric vacuum families of the heterotic string; essentially, we impose the special properties of the dilaton superfield S in an otherwise generic EQFT. We show that at the one-loop level, the dilaton dependence of the effective gauge couplings is completely universal: In terms of eq. (1.2),

$$g_{\mathrm{string}}^{-2} \;=\; \mathrm{Re}\, S \;+\; \frac{1}{16\pi^2}\, \Delta^{\mathrm{univ}}(M, \overline{M}), \tag{1.4}$$

M_{string} is $g_{\mathrm{string}} M_{\mathrm{Pl}}$ times a numerical constant and the gauge-group-specific threshold corrections $\Delta_a(M, \overline{M})$ are dilaton-independent; this is exactly what one obtains from the direct string-loop expansion.[31]

Furthermore, perturbative consistency between the EQFT and the string theory requires the "universal" threshold correction Δ^{univ} in eq. (1.4) to have exactly

the same moduli dependence as the Green-Schwarz term in the Kähler function, which is the field-theoretical description of the mixing between the dilaton and the moduli at the one-loop level of the string.[26−28] Consequently, given the functional form of the string-theoretical threshold corrections $\Delta_a(M,\overline{M})$, field-theoretical techniques can use such data to determine both the Green-Schwarz term and the exact Wilsonian gauge couplings of the EQFT.[25−29,32] Indeed, we shall see that the non-harmonic part of the moduli dependence of the combined threshold corrections

$$\tilde{\Delta}_a \;=\; \Delta_a \;+\; k_a \Delta^{\text{univ}} \tag{1.5}$$

is completely fixed by the low-energy EQFT in terms of the tree-level Kähler function. Thus, the discrepancy between the non-harmonic parts of the string-theoretical $\Delta_a(M,\overline{M})$ and the field-theoretical constraints on the $\tilde{\Delta}_a(M,\overline{M})$ determines Δ^{univ} and hence the Green-Schwarz term up to a holomorphic ambiguity. At the same time, the remaining, harmonic part of a $\tilde{\Delta}_a(M,\overline{M})$ determines the moduli dependence of the one-loop correction to the Wilsonian gauge coupling $f_a(S,M)$, and because of the no-renormalization theorem for the Wilsonian gauge couplings, the resulting f_a are exact to all orders of the perturbation theory.

In section 3 we demonstrate the power of the field-theoretical constraints on the holomorphic functions $f_a(M)$ and show that their exact form can often be obtained from essentially tree-level properties of the string vacua. Specifically, we consider families of factorizable $(0,2)$ orbifolds, which are invariant under a group of discrete symmetries called *modular transformations*. These symmetries are exact to all orders of the string perturbation theory, but the way they act on the massless charged fields can be fully determined from the tree-level Kähler function of the EQFT. In Appendix B, we present a string-theoretical calculation of the relevant parameters of this function, which in turn tells us the exact anomalous transformation rules for the Wilsonian gauge couplings f_a. Furthermore, we show that the holomorphic functions $f_a(M)$ have no singularities for any finite

values of the moduli fields and that their divergences in the decompactification limit are no worse than power-like with respect to the radii of the internal six-torus. Together, these data are sufficient to determine the functions $f_a(M)$ up to moduli-independent constants.

For the factorizable orbifolds we thus have two independent means of calculating the moduli dependence of the gauge couplings in an analytic form: The field-theoretical method outlined above, and also the direct string-theoretical approach of ref. 15. We show that the two calculations yield the same functional form for the moduli-dependent threshold corrections, but the numerical coefficients of similar terms are given by apparently unrelated formulæ. Nevertheless, for all the orbifolds we have studied we found those numerical coefficients to fully agree with each other; a number of examples are presented in Appendix C.

Finally, in section 4 of this article we discuss the $(2,2)$ vacuum families of the heterotic string; Calabi-Yau compactifications[33] are the best-known examples of such vacua. The $(2,2)$ families possess intricate tree-level relations between the couplings of the charged matter fields and the geometry of the moduli space.[34,35] These relations allow us to derive the anomalous transformation rules for the Wilsonian gauge couplings under discrete symmetries from the Kähler function and the transformation rules for the moduli fields, without any additional string-theoretical information. Furthermore, the gauge group of a generic $(2,2)$ vacuum is $E_6 \times E_8$, and the difference between the gauge couplings for E_6 and E_8 turns out to be related to the topological index F_1 defined in ref. 36. This index is computable in geometrical terms for the large-radius Calabi-Yau threefolds and thus provides additional information about the large-radius behavior of the gauge couplings. Consequently, given all the symmetries and all the singularities of a threefold's moduli space, one often has enough constraints for the holomorphic functions $f_6(M)$ and $f_8(M)$ to completely determine their form. As an example of this method, we calculate the dependence of the Wilsonian gauge couplings on the only $(1,1)$ modulus of the quintic threefold.[37,36] Alas, we are unable to compare

this field-theoretical result to a direct string-theoretical calculation because the moduli dependence of the superheavy particles' masses is not presently known for the quintic threefold. Since the same is true for most other currently known string vacua, the field-theoretical method of analysis appears to be indispensable.

2. Effective Quantum Field Theory of Generic String Vacua

2.1. Dilaton Dependence of the Wilsonian Couplings.

At energies below the Planck scale, all particle interactions can be described in terms of an Effective Quantum Field Theory (EQFT) for the light modes of the string. Local supersymmetry imposes severe constraints on the action of the EQFT; in particular, all interactions with at most two derivatives can be summarized in terms of the Kähler function $\mathcal{K}$, the superpotential W and the field-dependent gauge couplings f_a. It is important to distinguish between the Wilsonian couplings of an EQFT, which are coefficients of local quantum operators comprising the action of the theory, and between the momentum-dependent effective couplings that parametrize the scattering amplitudes.[21,13] For EQFTs quantized and cut-off in a manifestly locally supersymmetric fashion, the *Wilsonian* superpotential W and the *Wilsonian* gauge couplings f_a are holomorphic functions of the chiral superfields. Furthermore, W does not renormalize perturbatively while the renormalization of f_a is exhausted at the one-loop level of the perturbation theory. On the other hand, the effective gauge couplings $g_a^{-2}(p^2)^\star$ renormalize in all orders of the perturbation theory and their dependence on the moduli scalars is non-holomorphic.

The issue of manifestly locally supersymmetric quantization and regularization of EQFTs is discussed in detail in ref. 13. For the purposes of this article, let us iterate a few points: First, the supersymmetric cutoff discussed in ref. 13 is purely perturbative in nature and cannot be used to *define* a locally supersymmetric EQFT in a manifestly unitary non-perturbative way; there is no known supersymmetric analogue of the lattice cutoff for ordinary gauge theories. Therefore, our formalism presumes that all the Wilsonian couplings of the EQFT cut-off at the string threshold are weak enough to use perturbation theory; physically, this means that all the interactions *at the string scale* must be perturbatively weak. Note that this assumption does not exclude strong interactions at much lower energies. However, strong interactions right at the string scale would require a different field-theoretical formalism — as well as a non-perturbative string theory.

Second, manifest local supersymmetry of the regularized EQFT is not enough: One also needs to maintain full $d = 4$, $N = 1$ gauge invariance of the theory. (To be precise, the background gauge invariance should be manifest while the quantum gauge invariance is protected by the BRST symmetry.) Such a regularization ought to be possible, but the specific prescription displayed in ref. 13 presumes that only the gauge and the charged matter superfields are affected by the gauge transformations while the background gravitational and moduli superfields remain inert. In particular, we did not allow for linear superfields with Chern-Simons couplings to the gauge superfields because of technical difficulties with regularizing such couplings. Fortunately, linear superfields are always dual to chiral superfields, so one can avoid these difficulties by using the latter rather then the former.

Although from the field-theoretical point of view there is no harm (and much benefit) in putting all the scalar particles into chiral supermultiplets, from the string-theoretical point of view, using the chiral superfield S for the dilaton-axion-dilatino multiplet does it serious injustice. While for all other light particles the relation between the vertex operator of the string theory and the corresponding *unnormalized* quantum field of the EQFT is completely determined at the

⋆ In ref. 15 the effective gauge couplings were denoted $\{g_a(p^2)\}^{-2}$ in order to distinguish them from the Wilsonian couplings $(g_a^W)^{-2} = \operatorname{Re} f_a^W$. In this article, g_a always denote the effective gauge couplings while f_a always denote the Wilsonian couplings.

tree level and suffers from no corrections at higher orders, the dilaton, axion and dilatino vertices have similarly fixed relation to components of the linear superfield L, but their relation to the components of the chiral superfield S has to be adjusted order-by-order in perturbation theory. For this reason, whenever the low-energy limit of the heterotic string is discussed in terms of the generating function (sometimes called "the effective classical Lagrangian"), the linear superfield L gives a clearer picture of the dilaton-axion physics than the chiral superfield S.[26–29] On the other hand, the analytic properties of the Wilsonian couplings are more transparent in the chiral superfield formalism. Hence, for the purpose of this article, we prefer to work with S rather than L.

With all these preliminaries in mind, let us consider the Kähler function $\mathcal{K}$ of a string-based EQFT. $\mathcal{K}$ is a real analytic function of all the chiral superfields which controls their sigma-model-like interactions and the geometry of the field space. Generically, expanding $\mathcal{K}$ in powers of the matter superfields* Q^I and $\overline{Q}^{\bar{I}}$, we have

$$\mathcal{K}(\Phi,\overline{\Phi},Q,\overline{Q}) \;=\; \kappa^{-2}\,K(\Phi,\overline{\Phi}) \;+\; Z_{\bar{I}J}(\Phi,\overline{\Phi})\,\overline{Q}^{\bar{I}}e^{2V}Q^J \;+\;\cdots\,, \tag{2.1}$$

where Φ stands for all the chiral moduli superfields, including both the moduli M^i and the dilaton-axion S; the '$\cdots$' stand for the higher-order terms in Q^I which are irrelevant for the present discussion. Note that in our notations the matter superfields Q^I have canonical dimension one while the moduli are dimensionless. (A $\langle\Phi\rangle = O(1)$ corresponds to a Planck-sized modulus VEV in conventional units.) At the tree level of both the string theory and the EQFT,

$$K^{\rm tree}(\Phi,\overline{\Phi}) \;=\; -\log(S+\overline{S}) \;+\; \hat{K}(M,\overline{M}) \tag{2.2}$$

$\star$ In our terminology, the "matter" consists of all the scalar superfields that are not moduli and do not have Planck-sized VEVs. All the charged scalar superfields are matter, including the "hidden matter" charged under a "hidden" gauge symmetry. Gauge-singlet superfields that are prevented from acquiring Planck-sized VEVs by their Yukawa couplings are also treated as matter.

while the kinetic-energy matrix $Z_{\bar{I}J}$ for the matter fields depends on the string moduli M^i and $\overline{M}^{\bar{\imath}}$ but not on the dilaton. The specific form of the functions $\hat{K}(M,\overline{M})$ and $Z_{\bar{I}J}(M,\overline{M})$ depends on the details of the world-sheet SCFT defining a particular family of string vacua; their properties for orbifolds and Calabi-Yau manifolds are discussed later in this article (sections 3 and 4).

The dilaton and its superpartners arise in the spacetime sector of the world-sheet SCFT rather than in its internal sector. Consequently, its couplings are model independent at the tree level (*cf.* eqs. (1.3) and (2.2)), although the loop corrections destroy this universality. Thus, eq. (2.2) becomes

$$K(\Phi,\overline{\Phi}) \;=\; -\log(S+\overline{S}) \;+\; \hat{K}(M,\overline{M}) \;+\; \frac{V^{(1)}(M,\overline{M})}{8\pi^2(S+\overline{S})} \;+\; \frac{V^{(2)}(M,\overline{M})}{64\pi^4(S+\overline{S})^2} \;+\;\cdots; \tag{2.3}$$

similarly,

$$Z_{\bar{I}J}(\Phi,\overline{\Phi}) \;=\; Z^{(0)}_{\bar{I}J}(M,\overline{M}) \;+\; \frac{Z^{(1)}_{\bar{I}J}(M,\overline{M})}{8\pi^2(S+\overline{S})} \;+\; \frac{Z^{(2)}_{\bar{I}J}(M,\overline{M})}{64\pi^4(S+\overline{S})^2} \;+\;\cdots. \tag{2.4}$$

Note that in both formulæ, the dilaton appears only in combination $(S+\overline{S})$ — this is required by the continuous Peccei-Quinn symmetry $S\to S+i\gamma$, which holds to all orders of the perturbation theory. Furthermore, all the loop corrections come as power series in $1/8\pi^2(S+\overline{S})$, which serves as the string's coupling parameter.†

$\dagger$ Actually, the true string-loop counting parameter is $e^{-2\phi}$ where ϕ is the field defined by the dilaton's vertex operator; the precise relation between $e^{-2\phi}$ and $1/(16\pi^2\,{\rm Re}\,S)$ is itself subject to loop corrections. Therefore, the n-loop order corrections are generally of the order $O(1/(16\pi^2\,{\rm Re}\,S)^n)$, but they also contain sub-leading terms of higher orders in $1/(16\pi^2\,{\rm Re}\,S)$. For example, the one-loop effects not only determine the first-order coefficients $V^{(1)}(M,\overline{M})$ and $Z^{(1)}_{\bar{I}J}(M,\overline{M})$ in the series (2.3) and (2.4), but they also affect the higher order coefficients $V^{(2)}$ and $Z^{(2)}_{\bar{I}J}$, *etc., etc.* The way the loop counting works in terms of S is that the terms of the relative order $1/(16\pi^2\,{\rm Re}\,S)^n$ are completely determined at the n-loop order — not solely from the genus-n world sheet, but from all the genii from zero to n.

Now consider the superpotential W, which is a holomorphic function of the chiral superfields. Generically, it looks like

$$W(Q,\Phi)\ =\ \tfrac{1}{2}\mu_{IJ}(\Phi)\,Q^IQ^J\ +\ \tfrac{1}{3}Y_{IJK}(\Phi)\,Q^IQ^JQ^K\ +\ \cdots, \qquad (2.5)$$

where the '$\cdots$' stand for the non-renormalizable higher-order terms, but at the tree level of the heterotic string, the couplings μ_{IJ}, Y_{IJK}, *etc.*, depend only on the string moduli M^i but not on the dilaton S. In field theory, there is no renormalization of the Wilsonian superpotential‡ and even the finite threshold corrections to W are always completely determined at the tree level; in string theory, the same result follows from the Peccei-Quinn symmetry in combination with the holomorphicity.[39] Indeed, W is a holomorphic function of all the chiral superfields and thus cannot depend on the dilaton $\mathrm{Re}\,S$ without at the same time being dependent on the axion $\mathrm{Im}\,S$. On the other hand, the Peccei-Quinn symmetry does not allow for any non-derivative couplings of the axion field and hence to all orders of the perturbation theory, the entire Wilsonian superpotential (2.5) does not depend on the dilaton-axion superfield S. Furthermore, since the loop expansion of the string theory is controlled by the dilaton, it follows that the string-loop corrections do not affect the Wilsonian superpotential of the EQFT.

Like the superpotential, the Wilsonian gauge couplings f_a are holomorphic functions of the chiral superfields. Therefore, their dependence on the dilaton superfield S is also severely restricted by the Peccei-Quinn symmetry. Taking into account the tree-level formulæ (1.3) and the loop-counting property of the dilaton, it is easy to see that to all orders of the perturbation theory, one must have[22,40]

$$f_a(S,M)\ =\ k_a\cdot S\ +\ \tfrac{1}{16\pi^2}f_a^{(1)}(M)\ , \qquad (2.6)$$

where the second term on the right hand side is completely determined at the one-loop level of the perturbation theory and suffers from no higher-order corrections.

‡ The two-loop corrections discussed in ref. 38 affect the *effective* Yukawa couplings of a theory with massless chiral superfields but not its *Wilsonian* Yukawa couplings.

Again, we see that the string theory upholds the no-renormalization theorems of the supersymmetric field theory, where the *Wilsonian* gauge couplings do not suffer from either infinite or finite corrections beyond one loop.[21] We also see that the entire moduli dependence of the f_a is controlled by the one-loop effects; this result is fundamental for the present investigation.

Note that the stringy no-renormalization theorems for the superpotential and for the Wilsonian gauge couplings depend on the anomalous Peccei-Quinn symmetry and thus are purely perturbative in nature. Non-perturbatively, the continuous Peccei-Quinn symmetry is broken down to its anomaly-free discrete subgroup, whose invariants include the holomorphic exponential $\exp(-8\pi^2 S)$ of the dilaton superfield. Thus, beyond the perturbation theory, one expects

$$W(Q,M,S)\ =\ W_{\mathrm{tree}}(Q,M)\ +\ W_{\mathrm{NP}}\big(Q,M,e^{-8\pi^2 S}\big) \qquad (2.7)$$

and similar corrections to the Wilsonian gauge couplings $f_a(S,M)$.

2.2. The Effective Gauge Couplings and their Dilaton Dependence.

An effective quantum field theory has two kinds of gauge couplings (as well as Yukawa couplings, *etc.*): The momentum-dependent effective gauge couplings $g_a(p^2)$, which are directly related to physical quantities such as scattering amplitudes, and the Wilsonian gauge couplings f_a, which have no such direct physical meaning but rather serve as the input parameters of the EQFT. From the renormalization theory's point of view, the Wilsonian couplings are couplings of the EQFT from which the high-energy degrees of freedom are integrated out but the low-energy quantum operators are left as they are. On the other hand, the effective couplings account for all the quantum effects, both high-energy and low-energy. In the previous section, we discussed the Wilsonian gauge couplings

whose moduli- and dilaton-dependence is severely constrained by the holomorphicity and by the no-renormalization theorems. Let us now turn our attention to the effective gauge couplings $g_a(p^2)$.

In general, one calculates the effective couplings of an EQFT order-by-order in the perturbative expansion, by summing up the appropriate Feynman graphs of the regularized theory. According to Shifman and Vainshtein,[21] the effective gauge couplings of a rigidly-supersymmetric gauge theory can be calculated exactly in terms of the Wilsonian gauge couplings and the effective normalization matrix $Z^{\rm eff}_{\bar{I}J}(p^2)$ for the charged matter superfields.⋆ In ref. 13, we extended their formula to the locally supersymmetric EQFTs, in which

$$\begin{aligned} g_a^{-2}(p^2) \;&=\; \mathrm{Re}\, f_a \;+\; \frac{\mathbf{b}_a}{16\pi^2}\,\log\frac{\Lambda^2}{p^2} \\ &+\; \frac{\mathbf{c}_a}{16\pi^2}\,K \;+\; \frac{T(G_a)}{8\pi^2}\,\log g_a^{-2}(p^2) \;-\sum_r \frac{T_a(r)}{8\pi^2}\,\log\det Z^{\rm eff}_{(r)}(p^2), \end{aligned} \tag{2.8}$$

where r runs over the representations of the gauge group $G=\prod_a G_a$,

$$\begin{aligned} T_a(r) \;&=\; \mathrm{Tr}_r(T^2_{(a)}) \quad \text{for} \quad T_{(a)}\in G_a, \\ T(G_a) \;&=\; T_a(\text{adjoint of } G_a), \\ \mathbf{b}_a \;&=\sum_r n_r T_a(r) \;-\; 3T(G_a), \\ \mathbf{c}_a \;&=\sum_r n_r T_a(r) \;-\; T(G_a). \end{aligned} \tag{2.9}$$

n_r is the number of the matter multiplets in the representation r and $Z^{\rm eff}_{(r)}$ is the block of the effective normalization matrix $Z^{\rm eff}_{\bar{I}J}$ referring to the "flavor" indices of those matter multiplets. Finally, Λ is the nominal UV cutoff scale of the regularized EQFT; to assure that the functions $f_a(\Phi)$ correctly represent the

⋆ The effective normalization matrix $Z^{\rm eff}_{\bar{I}J}(p^2)$ itself is obtained from the perturbative 1PI 2-point Green's functions for the matter superfields $\overline{Q}^{\bar{I}}$ and Q^J.

moduli dependence of the Wilsonian gauge couplings, Λ must be independent of all the moduli. To be precise from the supergravitational point of view, Λ is constant in Planck units; thus, without loss of generality, we may set $\Lambda = M_{\rm Pl}$.

Eq. (2.8) holds to all orders of the perturbation theory, but actual evaluation of the effective gauge couplings in string based EQFTs to the n-loop order requires knowing the Kähler function K and the $Z^{\rm eff}_{\bar{I}J}$ to the $(n-1)$-loop order. Hence, beyond the one-loop level, analytic study of the moduli dependence of the effective gauge couplings is rather difficult, but at the one-loop level, we can use the tree-level approximations for the terms on the second line of eq. (2.8). Thus, inserting eqs. (2.2) and (2.6) into (2.8), we learn that the moduli and dilaton dependence of the gauge couplings for any $N=1$ heterotic string vacuum is given by

$$\begin{aligned} g_a^{-2}(p^2;\Phi,\overline{\Phi})^{\text{1-loop}} \;&=\; k_a\,\mathrm{Re}\,S \;+\; \frac{\mathbf{b}_a}{16\pi^2}\left(\log\frac{M^2_{\rm Pl}}{p^2} \;-\; \log(S+\overline{S})\right) \\ &+\; \frac{1}{16\pi^2}\left[\mathrm{Re}\, f_a^{(1)}(M) \;+\; \mathbf{c}_a\hat{K}(M,\overline{M}) \;-\sum_r 2T_a(r)\,\log\det Z^{\rm tree}_{(r)}(M,\overline{M})\right] \end{aligned} \tag{2.10}$$

plus a numerical constant of the order $O(1/16\pi^2)$. (In this article we study the field dependence of the gauge couplings and disregard any constant terms. However, such terms are important in determining the unification properties of the gauge couplings.)

There are two dilaton-dependent terms on the right hand side of eq. (2.10): The $k_a\,\mathrm{Re}\,S$ term, which is the tree-level coupling, and the $\log(S+\overline{S})$ term, which arises at the one-loop level. Remarkably, for any gauge coupling of any string-based $N=1$ supersymmetric EQFT, the coefficient of the latter one-loop term is precisely the coefficient $\mathbf{b}_a/16\pi^2$ of the one-loop beta-function. Hence, the dilaton dependence of the one-loop corrections to the effective gauge couplings amounts to a universal change of the couplings' unification scale: The natural starting point for the renormalization of the *effective* gauge couplings is not the

Planck scale but rather $M_{\rm Pl}/\sqrt{S+\overline{S}} \sim g_{\rm string}M_{\rm Pl} \sim (\alpha')^{-1/2} \sim M_{\rm string}$.[1]† On the other hand, the Wilsonian gauge couplings (2.6) unify at the Planck scale rather than at the string scale.

Actually, this "disagreement" as to whether the string threshold is at $M_{\rm string}$ or at $M_{\rm Pl}$ is similar to what happens at any threshold of a supersymmetric EQFT and has nothing specifically stringy about it. Indeed, at in ordinary GUT or at an intermediate-scale threshold, the effective gauge couplings of the unbroken part of the gauge group measure the threshold scale in terms of the physical masses of the heavy gauge bosons and other charged particles.[3] On the other hand, the Wilsonian gauge couplings are sensitive to the *unnormalized* Higgs VEVs[13] rather than to the physical masses; furthermore, it is the unnormalized Higgs VEVs that control the residual, non-renormalizable "weak" interactions due to the broken part of the gauge group. Thus, for every threshold we have two distinct threshold scales, which differ from each other by a factor proportional to the gauge coupling. This is precisely what happens at the string threshold: The physical masses of the massive string modes are proportional to $M_{\rm string}$ but the strengths of the non-renormalizable interactions below the string threshold, including the gravitational coupling κ, are proportional to powers of $g_{\rm string}/M_{\rm string} \sim 1/M_{\rm Pl}$ rather than simply powers of $1/M_{\rm string}$. Furthermore, whenever some particles can be either light or heavy depending on some moduli VEVs, in order for the physical masses of those particles to be of the order $M_{\rm string}$, the *unnormalized* VEVs of the Higgs-like moduli have to be of the order $O(M_{\rm Pl})$. Thus, the effective gauge couplings "feel" the string threshold at $M_{\rm string}$, while the Wilsonian couplings register that threshold at the Planck scale.

Thus far, we have discussed the effective gauge couplings g_a from the point of view of the string-based EQFT. However, the same couplings can be calculated

† Notice that the dilaton dependence of this string scale is a matter of convention. From the effective supergravity point of view, the Planck scale is field-independent while $M_{\rm string} \propto (S+\overline{S})^{-1/2}$, but in the string theory, it is natural to use the string scale as a field-independent unit of mass while $M_{\rm Pl} \propto (S+\overline{S})^{+1/2}$.

directly in the perturbative string theory. At the one-loop level, the result can be generally expressed in terms of eqs. (1.2). Emphasizing the moduli- and dilaton-dependence of all the terms, we have[4]

$$g_a^{-2}(p^2;S,\overline{S},M,\overline{M}) \;=\; k_a g_{\rm string}^{-2}(S,\overline{S},M,\overline{M}) \;+\; \frac{\mathbf{b}_a}{16\pi^2}\,\log\frac{M_{\rm string}^2}{p^2} + \;\frac{\Delta_a(M,\overline{M})}{16\pi^2}, \tag{2.11}$$

where, at the required level of accuracy, $M_{\rm string}^2$ is indeed given by the $M_{\rm Pl}^2/(S+\overline{S})$ times a numerical constant. The dilaton-dependence of the "universal" coupling $g_{\rm string}$ follows from the usual loop-counting arguments: At the tree level, $g_{\rm string}^{-2} = {\rm Re}\,S$, while at the one-loop level we have eq. (1.4). The non-universal (*i.e.*, gauge group dependent) string-threshold corrections Δ_a follow from the spectrum of the massive modes of the heterotic string:[4]

$$\Delta_a \;=\int\limits_{\Gamma}\frac{d^2\tau}{\tau_2}\,\left(\mathcal{B}_a(\tau,\bar\tau)\,-\,\mathbf{b}_a\right), \tag{2.12}$$

where the domain of integration Γ is the fundamental domain for the modulus τ of the world-sheet torus and

$$\mathcal{B}_a \;=\; \frac{2}{|\eta(\tau)|^4}\sum_{\text{even}\,\mathbf{s}}(-)^{s_1+s_2}\,\frac{dZ_\Psi(\mathbf{s},\bar\tau)}{2\pi i\,d\bar\tau}\,{\rm Tr}_{s_1}\Bigl(\Bigl(T_{(a)}^2-\tfrac{k_a}{8\pi\tau_2}\Bigr)\,(-)^{s_2\overline{F}}q^{L-\frac{11}{12}}\bar q^{\bar L-\frac{3}{8}}\Bigr)_{\rm int}. \tag{2.13}$$

Here $T_{(a)}$ is a generator of the gauge group G_a, $q=e^{2\pi i\tau}$, $\mathbf{s}=(s_1,s_2)$ denotes the NSR boundary conditions for the fermions on the supersymmetric side of the world sheet, $\overline{F}$ is their fermion number and Z_Ψ is the partition function of a free complex Weyl fermion; the $-\frac{k_a}{8\pi\tau_2}$ term[17] is included for the sake of the modular invariance of the functions $\tau_2\mathcal{B}_a(\tau,\bar\tau)$.⋆ The trace in eq. (2.13) is taken over the internal $c=(22,9)$ sector of the world-sheet SCFT; in general, it depends on all

⋆ The combination $(T_{(a)}^2-\frac{k_a}{8\pi\tau_2})$ emerges when the properly regularized Kac-Moody current-current correlator $\langle J_{(a)}(\zeta)J_{(a)}(0)\rangle$ is integrated over the world sheet.[17] The $-\frac{k_a}{8\pi\tau_2}$ term is obviously universal with respect to the gauge couplings and thus can be dropped from

the moduli of that internal sector, but not on the dilaton (which originates in the spacetime sector of the SCFT). Consequently, the string-threshold corrections $\Delta_a(M,\overline{M})$ depend on the string moduli M^i and $\overline{M}^{\bar{\imath}}$ but not on the dilaton field S. Thus, we conclude that *in string theory, the dilaton dependence of the effective gauge couplings g_a is exactly as in the field-theoretical formula* (2.10).[31]

2.3. Moduli Dependence of the Gauge Couplings

Having discussed the dilaton dependence of the gauge couplings we now turn our attention to their dependence on the moduli M^i originating in the internal sector of the world-sheet SCFT. Let us compare the one-string-loop formula (2.11) for the effective gauge couplings g_a with the one-loop EQFT formula (2.10). We have already seen that the dilaton-dependent parts of the two formulæ agree with each other. To assure agreement between the S-independent but moduli-dependent parts of eqs. (2.11) and (2.10), we now need

$$\operatorname{Re} f_a^{(1)}(M) \;=\; \tilde{\Delta}_a(M,\overline{M}) \;-\; \mathbf{c}_a\hat{K}(M,\overline{M}) \;+\sum_r 2T_a(r)\,\log\det Z_{(r)}^{\rm tree}(M,\overline{M}) \tag{2.14}$$

(up to an $O(1)$ numerical constant), where $\tilde{\Delta}_a$ are as in eq. (1.5). The obvious meaning of eqs. (2.14) is that they are formulæ for the one-loop corrections to the *Wilsonian* gauge couplings f_a in terms of quantities computable in string theory. The first term on the right hand side originates at the one-loop level of the heterotic string theory; the other two terms subtract the one-loop corrections arising in the low-energy EQFT and thus are computable in terms of the tree-level properties of the string. Despite the one-loop-approximate nature of these right-hand terms, the left-hand side is protected from any higher-loop corrections.

eq. (2.13) while its effect is absorbed into $\Delta^{\rm univ}$ — this is exactly what was done in ref. 4. However, it is not the only universal threshold correction to all the gauge couplings, so in addition to the $-\frac{k_a}{8\pi\tau_2}$ term in eq. (2.13), we also retain the $\Delta^{\rm univ}$ term in eq. (1.4) in order to account for the other universal corrections. Later in this section, we will show that it is this latter $\Delta^{\rm univ}$ term which is related to the Green-Schwarz term $V^{(1)}$ in eq. (2.3).

Thus, *as far as the Wilsonian gauge couplings f_a are concerned, eqs.* (2.14) *are exact* to all orders of the perturbation theory.

Manifest supersymmetry of the low-energy EQFT's Wilsonian Lagrangian requires holomorphicity of the functions $f_a^{(1)}(M)$. At the same time, none of the terms on the right hand side of eqs. (2.14) is — or has any a priori reason to be — a harmonic function of the moduli M^i and $\overline{M}^{\bar{\imath}}$; mutual cancellation of their non-harmonic parts is a non-trivial constraint. Thus, eqs. (2.14) impose powerful supersymmetric consistency conditions on $\tilde{\Delta}_a$ in terms of the tree-level couplings:

$$\partial_M\partial_{\overline{M}}\,\tilde{\Delta}_a(M,\overline{M}) \;=\; \partial_M\partial_{\overline{M}}\Big(\mathbf{c}_a\hat{K}(M,\overline{M}) \;-\sum_r 2\,\mathrm{Tr}_a(r)\,\log\det Z_{(r)}^{\rm tree}(M,\overline{M})\Big)\,. \tag{2.15}$$

Eqs. (2.15) apply to all $(d=4, N=1)$–supersymmetric vacuum families of the heterotic string; in the following section, we shall verify that they indeed hold true for the orbifolds and for the large-radius Calabi-Yau manifolds.

Eqs. (2.15) follow from the requirement of describing the physics of energies below the string threshold in terms of a locally supersymmetric EQFT. There are other constraints that follow from supersymmetry of the S-matrix and of the Green's functions without any reference to an EQFT. In particular, we argued in ref. 15 that the moduli dependence of the effective gauge couplings $g_a^{-2}(\Phi,\overline{\Phi})$ is related to the effective axionic couplings of the moduli scalars: The CP-odd part of the Green's function for two gauge bosons of G_a and a modulus scalar Φ^i is proportional to $\partial g_a^{-2}/\partial\Phi^i$. This relation must hold true in either field theory or string theory, and we shall use it momentarily to relate the universal part $\Delta^{\rm univ}$ of the threshold corrections (1.5) to the one-loop mixing (2.3) of the dilaton superfield S with the moduli superfields M^i.

For generic vacuum families of the heterotic string, the CP-odd one-string-loop scattering amplitudes were calculated by Antoniadis, Gava and Narain.[17]

They found

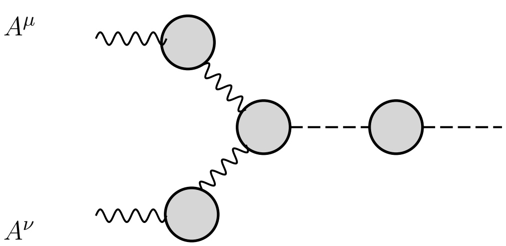
$$\begin{aligned}\mathcal{A}^-(A^\mu, A^\nu, M^i) &= \frac{+i\epsilon_{\mu\alpha\nu\beta}p_1^\alpha p_2^\beta}{16\pi^2} \times \frac{\partial\Delta_a}{\partial M^i}\,,\\ \mathcal{A}^-(A^\mu, A^\nu, \overline{M^i}) &= \frac{-i\epsilon_{\mu\alpha\nu\beta}p_1^\alpha p_2^\beta}{16\pi^2} \times \frac{\partial\Delta_a}{\partial \overline{M^i}}\,,\end{aligned} \tag{2.16}$$

where $\Delta_a(M, \overline{M})$ are given by eq. (2.12) in which $\mathcal{B}_a(\tau, \bar{\tau})$ of eq. (2.13) are replaced with

$$\mathcal{B}_a^{\rm AGN}(\tau, \bar{\tau}) = \frac{-1}{\eta^2(\tau)}\,\mathrm{Tr}_{\bar{R}}\left((-)^{\overline{F}-\frac{3}{2}}\,\overline{F}\,\left(T_{(a)}^2 - \tfrac{k_a}{8\pi\tau_2}\right) q^{L-\frac{11}{12}}\,\bar{q}^{\bar{L}-\frac{3}{8}}\right)_{\rm int}, \tag{2.17}$$

involving the odd Ramond-Ramond sector rather than the 3 even sectors.[⋆] Actually, for all spacetime-supersymmetric vacua $\mathcal{B}_a^{\rm AGN} = \mathcal{B}_a$, — this is one of the Riemann identities between characters of different NSR sectors of $(0,2)$ superconformal algebras; this particular identity is proven in Appendix A.

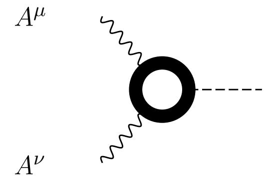
There is a subtle difference between the string-theoretical and the field-theoretical axionic couplings of the moduli: In field theory, the axionic couplings related by the spacetime supersymmetry to the derivatives of the the effective gauge couplings (2.11) are 1PI Green's functions, but the string-theoretical axionic amplitudes (2.16) are fully-dressed scattering amplitudes. Diagrammatically, the relation between these fully-dressed amplitudes and the 1PI Green's functions is



(2.18)

⋆ The right-moving fermion number operator $\overline{F}$ has half-integer values in the Ramond sector, hence the $(-1)^{\overline{F}-\frac{3}{2}}$ sign factor for the Ramond-Ramond boundary conditions. However, by abuse of notations, this factor is commonly written as simply $(-1)^{\overline{F}}$.

which at the one-loop level reduces to



M or

(2.19)

Note that the virtual particle in the second diagram here has to be the axion $\mathrm{Im}\,S$ because it is the only scalar with tree-level axionic couplings to the gauge bosons. At the one-loop level, the mixing of the axion $\mathrm{Im}\,S$ with the moduli M^i and $\overline{M^i}$ is controlled by the "Green-Schwarz" term $V^{(1)}(M, \overline{M})$ (*cf.* eq. (2.3)); thus, in light of eqs. (2.11) and (1.4),

$$\begin{aligned}\mathcal{A}^-(A^\mu, A^\nu, M^i) &= +i\epsilon_{\mu\alpha\nu\beta}p_1^\alpha p_2^\beta \left(\frac{\partial g_a^{-2}}{\partial M^i} + \frac{k_a}{16\pi^2}\frac{\partial V^{(1)}}{\partial M^i}\right)\\ &= \frac{+i\epsilon_{\mu\alpha\nu\beta}p_1^\alpha p_2^\beta}{16\pi^2} \times \frac{\partial}{\partial M^i}\left(\Delta_a + k_a\Delta^{\rm univ} + k_a V^{(1)}\right),\\ \mathcal{A}^-(A^\mu, A^\nu, \overline{M^i}) &= -i\epsilon_{\mu\alpha\nu\beta}p_1^\alpha p_2^\beta \left(\frac{\partial g_a^{-2}}{\partial \overline{M^i}} + \frac{k_a}{16\pi^2}\frac{\partial V^{(1)}}{\partial \overline{M^i}}\right)\\ &= \frac{-i\epsilon_{\mu\alpha\nu\beta}p_1^\alpha p_2^\beta}{16\pi^2} \times \frac{\partial}{\partial \overline{M^i}}\left(\Delta_a + k_a\Delta^{\rm univ} + k_a V^{(1)}\right).\end{aligned} \tag{2.20}$$

Comparing *these* amplitudes with the string amplitudes (2.16), we now conclude that the spacetime supersymmetry is indeed consistent, provided

$$\Delta^{\rm univ}(M, \overline{M}) = -V^{(1)}(M, \overline{M}) \tag{2.21}$$

M or $\overline{M}$,

(up to a moduli-independent constant). In other words, at the one-loop level, $g_{\rm string}^{-2} = \mathrm{Re}\,S - (1/8\pi^2)V^{(1)}(M, \overline{M}) + \cdots$, in precise agreement with the definition of the universal string coupling in the linear multiplet formalism.[26−29,17,41]

Ref. 17 gives an explicit string-theoretical formula for the Green-Schwarz function $V^{(1)}(M, \overline{M})$, but the supersymmetric consistency conditions (2.15) allow

us to obtain this function without any additional string-theoretical calculations. Indeed, eqs. (2.15), (1.5) and (2.21) together imply

$$\partial_M \partial_{\overline{M}} V^{(1)} \ = \ k_a^{-1} \, \partial_M \partial_{\overline{M}} \left(\Delta_a \ - \ \mathbf{c}_a \hat{K} \ - \sum_r 2 \, \mathrm{Tr}_a(r) \, \log\det Z^{\mathrm{tree}}_{(r)} \right), \tag{2.22}$$

where $\hat{K}$ and $Z^{\mathrm{tree}}_{(r)}$ are determined at the tree level of the string theory while Δ_a is computed via eqs. (2.12) and (2.13) (or (2.17)). Eq. (2.22) determines the Green-Schwarz function $V^{(1)}(M, \overline{M})$ up to an arbitrary harmonic function $H(M)$,

$$V^{(1)}(M, \overline{M}) \ \to \ V^{(1)}(M, \overline{M}) \ + \ H(M) \ + \ H^*(\overline{M}). \tag{2.23}$$

This remaining indeterminacy is related to a fact that unlike all other fields of the low-energy EQFT, the chiral dilaton superfield S has no fixed relation to vertices of the string theory. Thus, we are free to re-define

$$S \ \to \ S \ + \ \tfrac{1}{8\pi^2} \, H(M) \tag{2.24}$$

as long as H is a holomorphic function of the chiral moduli M^i. This redefinition naturally affects the analytic form of the Kähler function (2.3); at the one-loop level, the effect is precisely (2.23).

3. Field Theoretical Constraints for Orbifolds

Thus far our discussion of the moduli dependent gauge couplings was completely generic; we gave a general formula for the moduli and dilaton dependence of the gauge couplings (eqs. (2.10)) and outlined how to compute this moduli dependence in string theory (eqs. (2.12), (2.17) and (2.22)). However, for many families of the heterotic string vacua, their special properties may be used to severely constrain the holomorphic functions $f_a^{(1)}(M)$ and sometimes determine their exact analytic forms (up to a constant) without performing any string-loop

calculations. There are several sources of such constraints: The geometry of the moduli space of a vacuum family controls the locations of all the singular points (or subspaces) of the functions $f_a^{(1)}(M)$ and the types of the respective singularities. Perturbative consistency of the EQFT imposes limits on the growth of the f_a along asymptotic directions such as infinite radius. Finally, vacuum families often have exact discrete symmetries, which also impose constraints on the analytic form of the $f_a(M)$.

The effective gauge couplings $g_a(p^2)$ are physical quantities and hence must remain invariant under all the exact symmetries of the theory. The transformation properties of the Wilsonian gauge couplings f_a are not so obvious because these couplings act as counterterms cancelling potential anomalies of the EQFT.[25–27,13] Specifically, there are two supersymmetrized Adler-Bell-Jackiw anomalies at play: The Konishi anomaly arising when charged chiral matter superfields are mixed with each other,[42]

$$Q^I \ \to \ \Upsilon^I{}_J(\Phi) \, Q^J \ + \ O(Q^2/M_{\mathrm{Pl}}), \tag{3.1}$$

and the anomaly of the Kähler transformations

$$K \ \to \ K \ + \ \mathcal{J}(\Phi) \ + \ \mathcal{J}^*(\overline{\Phi}), \qquad W \ \to \ W \times \exp(-\mathcal{J}(\Phi)); \tag{3.2}$$

cancellation of the combined anomaly requires[13]

$$f_a \ \to \ f_a \ - \ \frac{\mathbf{c}_a}{8\pi^2} \mathcal{J}(\Phi) \ - \sum_r \frac{T_a(r)}{4\pi^2} \, \log\det \Upsilon^{(r)}(\Phi). \tag{3.3}$$

In this section we are going to demonstrate the power of the field-theoretical constraints on the Wilsonian gauge couplings. We shall see that for many orbifolds vacua of the heterotic string[43] such constraints uniquely determine the functions $f_a^{(1)}(M)$; furthermore, when we restate the results in terms of the moduli dependence of the physical threshold corrections $\tilde{\Delta}_a$, we shall find the latter in

complete agreement with the string-theoretical threshold corrections computed in ref. 15. To facilitate this comparison, we focus on exactly the same class of factorizable abelian $(0,2)$ orbifolds as were discussed in ref. 15, although we briefly return to more general orbifolds at the end of this section.

The best way to describe a generic factorizable orbifold is to build one. As usual, we begin with a toroidal compactification of the ten-dimensional heterotic string. At this stage, we do not allow any Wilson lines, discrete or continuous; instead, we keep the six internal dimensions completely separate from the $E_8 \times E_8$ degrees of freedom. Moreover, we split the six internal dimensions into three orthogonal planes and compactify each plane into a separate two-torus. The purpose of this restriction is to simplify the moduli space of the theory: as long as we keep the above constrains, we have two complex moduli T^i and U^i for each of the three planes ($i = 1, 2, 3$); furthermore, there is a separate $SL(2, \mathbf{Z})$ duality symmetry for each of the six moduli T^i and U^i.[44]

At the second stage, we twist the theory by a discrete symmetry group; this may require freezing of some or all of the U^i moduli. We insists that all the group elements avoid mixing the planes but rotate each plane onto itself; together with the need to preserve $N = 1$ spacetime supersymmetry, this requirement limits us to the abelian twist groups only. Finally, we limit the asymmetry of the twists by requiring all rotations of the three internal planes to be symmetric with respect to the left-moving and right-moving bosonic operators comprising each plane. However, beyond the constraints of modular invariance, we do not ask for any relation between the twisting of fermionic superpartners ψ^i of the internal coordinates X^i and the way the same twist acts upon the $E_8 \times E_8$ degrees of freedom. Thus, we are not limited to the completely symmetric $(2,2)$ orbifolds but allow for a rather large class of the $(0,2)$ orbifolds, and our analysis of the moduli dependent gauge couplings applies to all the gauge symmetries such an orbifold might have.

In the literature, the term "modulus" has an ambiguous meaning in the orbifold context. Here we consider an exact flat direction of the scalar potential to be a modulus if and only if one can vary its expectation value without changing the spectrum of the light fields that appear in the low-energy EQFT. In particular, all the moduli must be neutral with respect to all the low-energy gauge symmetries. For the purposes of this article, we keep in the low-energy EQFT all the gauge symmetries originating in the $E_8 \times E_8$ (or D_{16}) world-sheet degrees of freedom. A twisted state of an abelian orbifold is generally charged under at least one of those gauge symmetries and thus should not be considered a modulus even if it happens to parametrize an exactly flat direction of the scalar potential. Disregarding possible exceptions to this rule,⋆ we limit the present analysis of the moduli-dependent couplings to the untwisted, *i.e.*, toroidal moduli of factorizable orbifolds. Furthermore, for the sake of notational simplicity, we concentrate on the diagonal moduli T^i and U^i of the tree two-tori, although our analysis could be straightforwardly generalized to include the off-diagonal toroidal moduli of the Z_3, Z_4 and Z_6' orbifolds.

On the other hand, treating all the T^i and U^i as moduli means that we may only consider the couplings of the gauge symmetries that originate in the $E_8 \times E_8$ (or the D_{16}) and thus remain unbroken for generic values of the T^i and U^i. For some special values of these moduli, the momenta/windings of the six-torus give rise to additional massless vector bosons[43], but the couplings associated with such "accidental" gauge symmetries do not belong in the low-energy EQFT which treats all of the T^i and U^i as moduli. Instead, they belong in the EQFT that includes the "accidentally" light particles and re-interprets some of the T^i and U^i as charged Higgs fields rather than moduli. However, in this article we only concentrate on the more generic gauge couplings.

⋆ If a $(0,2)$ orbifold has sectors where the X^i (and the ψ^i) are twisted but the $E_8 \times E_8$ (or D_{16}) degrees of freedom remain completely untwisted, then such sectors may give rise to neutral massless scalars. It is not known if any such neutral twisted scalars have exactly flat potentials; if they do, they should be regarded as moduli, but we are not going to discuss them any further in this article.

Having defined the factorizable orbifolds and their gauge couplings, let us now consider the symmetry constraints on the f_a. Under the modular symmetries, the toroidal moduli of an orbifold transform according to[44]

$$M \;\to\; \frac{aM-ib}{icM+d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}), \tag{3.4}$$

i.e., $a,b,c,d\in\mathbf{Z}$ and $ad-bc=1$; the M here is either a T^i or a U^i and there is a separate $SL(2,\mathbf{Z})$ matrix for each such modulus. The tree-level Kähler function for the toroidal moduli of any factorizable orbifold has the form[14,45,34]

$$\hat{K}(M,\overline{M}) \;=\; -\sum_i \log(M^i+\overline{M}^i) \tag{3.5}$$

where the sum is over all the toroidal moduli; under the symmetries (3.4), this Kähler function transforms according to eq. (3.2) with

$$\mathcal{J}(M) \;=\; \sum_i \log(ic_iM^i+d_i). \tag{3.6}$$

Note that although the tree-level Kähler function (3.5) is corrected by string loops, the holomorphic function (3.6) has to be exact to all orders of the perturbation theory: This follows from the fact that the Wilsonian superpotential W in eq. (3.2) is protected from any perturbative renormalization.

The same argument can be used to obtain the exact transformation properties of the matter fields themselves from the tree-level matter normalization matrix $Z^{(0)}_{I\bar{J}}(M,\overline{M})$. This matrix has to be calculated directly from the string theory; besides the spectrum of the light particles, it is the only model-dependent string-theoretical data we need for our purposes. The calculation is performed in Appendix B (see also refs. 34, 32 and 17); the result is

$$Z^{(0)}_{I\bar{J}}(M,\overline{M}) \;=\; \delta_{I\bar{J}}\times\prod_i \left(M^i+\overline{M}^i\right)^{-q^i_I}, \tag{3.7}$$

where the exponents q^i_I are rational numbers depending on the twist sector giving rise to a matter field Q^I, on the angle by which the internal X^i coordinate is rotated by that twist and on the presence of the ∂X^i or $\partial\overline{X}^i$ world-sheet operators in the vertex for the Q^I. In Appendix B we give an explicit formula for all the q^i_I, but for the present discussion we only need the general form of eq. (3.7). The transformation of $Z^{(0)}_{I\bar{J}}$ under the $SL(2,\mathbf{Z})$ symmetries follow from eqs. (3.7) and (3.4) and in turn determine the transformation rules for the matter fields:

$$Q^I \;\to\; \widehat{\Upsilon}^I{}_J Q^J \times \prod_i (ic_iM^i+d_i)^{-q^i_I} \tag{3.8}$$

where $\widehat{\Upsilon}^I{}_J$ is a moduli-independent unitary matrix. Again, although the tree-level normalization matrix (3.7) suffers from both field-theoretical and string-theoretical higher order corrections, the transformation rules (3.8) are exact to all orders of the perturbation theory.

According to eq. (3.3), the modular transformation rules (3.6) and (3.8) completely determine the behavior of the Wilsonian gauge couplings f_a under the same modular transformations. Since the chiral dilaton superfield S is defined by the string only up to re-definitions (2.24), we adopt a convention in which S is completely inert under all the modular transformations. Hence, the entire transformation (3.3) is due to the Wilsonian threshold corrections $f^1_a(M)$, which thus transform according to⋆

$$f^1_a \;\to\; f^1_a \;-\sum_i 2\alpha^i_a\,\log(ic_iM^i+d_i) \tag{3.9}$$

(modulo an imaginary constant), where

$$\alpha^i_a \;=\sum_I T_a(Q^I)\,(1-2q^i_I) \;-\; T(G_a). \tag{3.10}$$

Mathematically, eqs (3.9) resemble the modular transformation rules for loga-

⋆ This also follows from eqs. (2.14).

rithm of the Dedekind's η function, which leads us to conclude that

$$f_a^1(M) \;=\; -\sum_i 4\alpha_a^i \,\log\eta(iM^i) \;+p_a(M), \tag{3.11}$$

where p_a are modular invariant (up to imaginary constants) holomorphic functions of the toroidal moduli M^i. This form makes manifest the modular invariance of the physical threshold corrections $\tilde{\Delta}_a$. Indeed, substituting eqs. (3.5), (3.7) and (3.11) into (2.14), we arrive at

$$\tilde{\Delta}_a(M,\overline{M}) \;=-\sum_i \alpha_a^i\,\log\left(\left|\eta(iM^i)\right|^4\,\mathrm{Re}\,M^i\right)\;+\;\mathrm{Re}\,p_a(M). \tag{3.12}$$

We are now going to argue that for factorizable orbifolds $p_a(M) = \mathrm{const}$; as a first step in this direction, let us consider possible singularities of these functions. As far as the perturbative string theory is concerned, a toroidal modulus of a factorizable orbifold is either completely frozen by the twist group or else can take any finite values in the right half of the complex plane ($\mathrm{Re}\,M > 0$). However, a perfectly regular string vacuum may lead to a singular EQFT if some particles that are massive for generic values of the moduli become massless at that particular point (or subspace) of the moduli space. In factorizable orbifolds, this happens whenever $T^i \equiv U^i \pmod{SL(2,\mathbf{Z})}$, at which point momenta/windings in the X^i plane give rise to several massless particles; however, such "accidentally massless" particles are always completely neutral with respect to any low-energy gauge symmetry originating in the $E_8\times E_8$ (or D_{16}). Hence, although some of the low-energy EQFT's couplings may become singular when $T^i \equiv U^i \pmod{SL(2,\mathbf{Z})}$, the gauge couplings we are interested in do not develop any singularities *at the one-loop level*.† At higher-loop levels, these couplings may also become singular, but the absence of the one-loop singularities is all we need to

† For a discussion of some of the singular couplings see ref. 46.

conclude that the *Wilsonian gauge couplings* f_a have no singularities anywhere in the moduli space. This argument exemplifies the power of the no-renormalization theorem for the Wilsonian gauge couplings f_a: They are completely determined at the one-loop level and thus do not care whether the non-harmonic terms have any higher-loop singularities.

The non-singular behavior of the Wilsonian gauge couplings implies that the $p_a(M)$ in eqs. (3.11) and (3.12) are holomorphic functions without any singularities for $\mathrm{Re}\,M > 0$ and they are also modular invariant (modulo imaginary constants) with respect to separate $SL(2,\mathbf{Z})$ transformations (3.4) for each modulus M^i. Mathematically, these two constraints imply that $p_a(M)$ are polynomials (or convergent power series) of $j(iM^i)$ where j is the $SL(2,\mathbf{Z})$ invariant function that maps the fundamental domain of the symmetry onto the complex sphere.[47] The $j(iM)$ function is finite for any finite M in the right half plane, but it grows exponentially in the $\mathrm{Re}\,M\to\infty$ limit. Hence, each p_a is either entirely independent of a modulus M^i or else it has to grow at least exponentially when $\mathrm{Re}\,M^i$ becomes large.

Now consider the physics of the $\mathrm{Re}\,M^i\to\infty$ limits: In the $\mathrm{Re}\,T^i\to\infty$ limit, both periods of the two-torus for the internal complex coordinate X^i become very large. In the $\mathrm{Re}\,U^i$ limit, one of the periods of the same two-torus becomes very large while the other period becomes very small; by duality, the physics of this limit is the same as if both periods were very large. Thus, in each of the $\mathrm{Re}\,M^i\to\infty$ limits, the orbifold decompactifies and the four-dimensional low-energy EQFT becomes rather singular. However, we will show momentarily that the singularity of such a limit is relatively mild; specifically, the gauge couplings do not grow larger than $O(\mathrm{Re}\,M^i)$.

In order to obtain this bound, let us consider the following double limit of an orbifold vacuum: $\mathrm{Re}\,T^1\to\infty$, other M^i fixed, $\mathrm{Re}\,S\to\infty$ while the ratio $\mathrm{Re}\,T^1/\,\mathrm{Re}\,S$ is kept finite and small. Since all the physical couplings of the four-dimensional EQFT are proportional to the negative powers of the dilaton $\mathrm{Re}\,S$,

they are so small in this limit that the effective theory below the compactification scale is essentially classical. Above the compactification scale $\sqrt{\alpha'/\operatorname{Re}T^1}$ of the complex coordinate X^1, the theory is effectively six-dimensional and its loop-counting parameter is no longer simply $g_4^2 \sim 1/\operatorname{Re}S$ but rather $g_6^2 \sim \operatorname{Re}T^1/\operatorname{Re}S$. This modified loop counting applies not just to the six-dimensional field theory but to the string theory as well.[48,49] Thus, as long as g_6^2 remains sufficiently small, we can use the perturbation theory at all energies. Physically, this implies that the loop corrections to the four-dimensional couplings should be small compared to their tree-level values. In particular, we got to have

$$|\tilde{\Delta}_a| \ll \operatorname{Re}S \quad \text{as long as } g_6^2 \ll 1,\ \textit{i.e.},\ ReS \gg \operatorname{Re}T^1 \tag{3.13}$$

(note that all the other moduli M^i are fixed here). Since the one-loop threshold corrections $\tilde{\Delta}_a$ do not depend on the dilaton, the inequality (3.13) is nothing but a bound on the large $\operatorname{Re}T^1$ limit of the $\tilde{\Delta}_a$, namely $|\tilde{\Delta}_a| \leq O(\operatorname{Re}T^1)$ in the large $\operatorname{Re}T^1$ limit. Naturally, similar bounds

$$|\tilde{\Delta}_a| \ \leq\ O(\operatorname{Re}M) \tag{3.14}$$

apply to all the other $\operatorname{Re}M^i \to \infty$ limits. As an immediate corollary of these bounds, we may finally eliminate the functions $p_a(M)$ in eqs. (3.11) and (3.12). Indeed,

$$-\alpha_a^i \log\bigl(|\eta(iM^i)|^4 \operatorname{Re}M^i\bigr) \ \approx\ \tfrac{\pi}{3}\alpha_a^i \operatorname{Re}M^i \quad \text{for large } \operatorname{Re}M^i, \tag{3.15}$$

in good agreement with the bounds (3.14). On the other hand, the $\operatorname{Re}p_a(M)$ terms in eq. (3.12) are either constant or else they grow exponentially or even faster with $\operatorname{Re}M^i \to \infty$. Having seen that the consistency of the perturbation theory does not allow for such a rapid growth, we conclude that p_a have to be moduli-independent constants.

This completes our field-theoretical study of factorizable orbifolds of the heterotic string. We have used the following string-theoretical data as input: The spectrum of the light particles, the *tree-level* couplings (3.5) and (3.7) and, most importantly, the knowledge that the $SL(2,\mathbf{Z})$ modular symmetries (3.4) are not merely symmetries of the tree-level couplings but exact symmetries of the vacuum states of the heterotic string.⋆ Given this string theoretical input, the field-theoretical constraints then determine the Wilsonian gauge couplings to be exactly

$$f_a(S,M) \ =\ k_a S\ -\sum_i \frac{\alpha_a^i}{4\pi^2}\log\eta(iM^i)\ +\ \text{const} \tag{3.16}$$

while the one-loop threshold corrections to the physical gauge couplings are precisely

$$\tilde{\Delta}_a(M,\overline{M})\ = -\sum_i \alpha_a^i \log\Bigl(\bigl|\eta(iM^i)\bigr|^4 \operatorname{Re}M^i\Bigr)\ +\ \text{const.} \tag{3.17}$$

In the large volume limit, these threshold corrections behave as

$$\tilde{\Delta}_a \ \approx\ \frac{\pi}{3}\sum_i \alpha_a^i \operatorname{Re}M^i\ \propto\ (\text{Radius})^2 \tag{3.18}$$

(*cf.* ref. 40); in the following section we shall see that for the smooth Calabi-Yau compactifications the threshold corrections also behave like the square of the radius in the large radius limit.

Having derived eqs. (3.17) from the field-theoretical arguments, let us now compare them to the threshold corrections that follow from the direct one-string-loop calculations of ref. 15. In string theory, the moduli dependence of the non-universal threshold corrections $\Delta_a(M,\overline{M})$ for an orbifold with a twist group D is

⋆ In general, when using a symmetry to restrict the form of the gauge (or other) couplings, it is extremely important to verify the symmetry at the string theoretical level since the tree-level low-derivative couplings often have apparent symmetries that do not survive string loop corrections. For example, the tree-level σ-model-like couplings (3.5) and (3.7) of factorizable orbifolds are invariant under continuous $SL(2,\mathbf{R})$ symmetries, but only the discrete $SL(2,\mathbf{Z})$ symmetries are true symmetries of the string theory.

determined not by that orbifold itself but rather by the related orbifolds where the same internal six-torus is twisted by the little subgroups D_i of the complex coordinates X^i. That is, D_i is comprised of the members of D that do not rotate the X^i; for example, for the Z_6 orbifold whose twist group D is generated by the rotation $\Theta=(e^{2\pi i/6},e^{2\pi i/3},-1)$, the little subgroup D_3 of the third complex plane is a Z_3 generated by the $\Theta^2=(e^{2\pi i/3},e^{4\pi i/3},1)$, the little subgroup D_2 of the second plane is a Z_2 generated by the $\Theta^3=(-1,+1,-1)$ and the little subgroup of the first plane is trivial. When the original D-orbifold has $N=1$ spacetime supersymmetry, the D_i-orbifolds are $N=2$ supersymmetric; they are also particularly simple when the original D-orbifold is factorizable, which is precisely why we have concentrated on the factorizable orbifolds in ref. 15.

Translating the main result of ref. 15 into the notations of the present paper, we have for any factorizable $(0,2)$ orbifold

$$\Delta_a(M,\overline{M}) = -\sum_{\substack{i=1,2,3\\ |D_i|>1}} \frac{\mathbf{b}_a^{N=2}(i)}{|D|/|D_i|}\left[\log\left(\left|\eta(iT^i)\right|^4 \operatorname{Re}T^i\right)+\log\left(\left|\eta(iU^i)\right|^4 \operatorname{Re}U^i\right)\right] + k_a \sum_{\substack{i=1,2,3\\ |D_i|>1}} \frac{\Omega(T^i,U^i)}{|D|/|D_i|} \; + \; \text{const}, \tag{3.19}$$

where $\mathbf{b}_a^{N=2}(i)$ are the β-function coefficients of the $N=2$ supersymmetric D_i-orbifold. The second sum in this formula constitutes an additional universal term, quite distinct from the $\Delta^{\rm univ}$ in eqs. (1.4) and (1.5); this term was not computed or even discussed in ref. 15 where we calculated only the differences between the threshold corrections for the different gauge couplings.

The functional form of the first sum in eq. (3.19) is exactly the same as that of the field-theoretical threshold corrections (3.17). Since the latter differ from the string-theoretical threshold corrections by the Green-Schwarz function

$V^{(1)}(M,\overline{M})$ (*cf.* eqs. (1.5) and (2.21)), we immediately conclude that

$$V^{(1)}(M,\overline{M}) \;=\; \sum_{i=1,2,3}\delta^i_{\rm GS}\left[\log\left(\left|\eta(iT^i)\right|^4 \operatorname{Re}T^i\right) + \log\left(\left|\eta(iU^i)\right|^4 \operatorname{Re}U^i\right)\right] + \sum_{\substack{i=1,2,3\\ |D_i|>1}} \frac{\Omega(T^i,U^i,\overline{T}^i,\overline{U}^i)}{|D|/|D_i|} \; + \; \text{const}, \tag{3.20}$$

where the $\delta^i_{\rm GS}$ are some numerical constants to which we shall return in a moment. The non-harmonic part of the first sum in this formula has the same form as the Green-Schwarz function discussed in refs. 26–28,17. The harmonic $\log|\eta(iM)|^4$ terms in this sum follow from our convention that the dilaton S (and hence the Green-Schwarz function V) should be inert under the $SL(2,\mathbf{Z})$ modular transformations (3.4). Without this convention, those harmonic terms can be re-defined away according to eqs. (2.23) and (2.24).

The second sum in eq. (3.20) and the role it plays in $N=2$ and $N=1$ orbifolds will be discussed in detail in a forthcoming article. For the present, we simply state without proof that the function $\Omega(T,U)$ is non-trivial, that it is universal for all the factorizable orbifolds and that it is finite but mildly singular when $T^i\equiv U^i \pmod{SL(2,\mathbf{Z})}$. The simplest way to describe this function is to say that Ω is the $[SL(2,\mathbf{Z})]^2$ modular invariant solution of the differential equations

$$\left[(T+\overline{T})^2\partial_T\partial_{\overline{T}}-2\right]\Omega \;=\; \left[(U+\overline{U})^2\partial_U\partial_{\overline{U}}-2\right]\Omega \;=\; \log\left|j(iT)-j(iU)\right|. \tag{3.21}$$

Turning our attention to the non-universal parts of the threshold corrections (3.17) and (3.19), we see that their functional form is the same but the coefficients do not seem to be related to each other. Thus, the field-theoretical and the string-theoretical threshold corrections agree with each other if and only if

$$\alpha^i_a \;=\; \frac{\mathbf{b}_a^{N=2}(i)}{|D|/|D_i|} \;+\; k_a\delta^i_{\rm GS} \tag{3.22}$$

where the constants $\delta^i_{\rm GS}$ are exactly as in eq. (3.20). In particular, for a plane X^i that is twisted by all the nontrivial members of the orbifold group D, the "$N=2$" D_i-orbifold is actually an untwisted six-torus with $N=4$ spacetime supersymmetry and zero $\mathbf{b}_a(i)$; for such a plane we should have $\alpha^i_a = k_a\delta^i_{\rm GS}$ for all the couplings. Eqs. (3.22) are just as essential for the perturbative consistency of the orbifold vacua of the heterotic string as the cancellation of the ordinary triangle anomalies for all the low-energy gauge symmetries; unfortunately, they are also just as difficult to prove directly from the string theory.

In Appendices B and C.1 we prove two general string-theoretical properties of the field-theoretical "modular anomaly" coefficients (3.10): First, whenever a factorizable orbifold has a $(1,2)$ modulus whose value is not frozen by the orbifold's twist group D, such modulus always has exactly the same α^i_a as the $(1,1)$ modulus of the same internal plane. This agrees with the behavior of the string-theoretical coefficients $\mathbf{b}^{N=2}_a(i)/(|D|/|D_i|)$ and also allows us not to distinguish between the T^i and the U^i in eqs. (3.22). Second, whenever a factorizable orbifold has $N=2$ spacetime supersymmetry because one of the three internal planes is never rotated by the orbifold's twist group, the coefficients α_a for that plane are indeed equal to the β-function coefficients $\mathbf{b}_a$ while for the other two planes $\alpha^i_a = \mathbf{b}_a(i) = 0$. Alas, in the $N=1$ case, we do not have a general string-theoretical proof of eqs. (3.22) but only eqs. (3.10) that give us the coefficients α^i_a for any particular orbifold. In a way of a numerical experiment, we have calculated the α^i_a for a few scores of orbifolds and found that they all satisfy eqs. (3.22). A dozen examples of such calculations are presented in Appendices C.2–4. (See also refs. 32 and 26.)

We conclude this section with a few words about the orbifolds whose internal six-tori do not factorize into products of separate two-tori for each of the three X^i. At the tree level, relaxing the factorizability condition does not affect in any way the Kähler functions (3.5) and (3.7); thus, we still have modular symmetries that act like (3.4) and (3.8) on the moduli and matter superfields

of the theory and under which the Wilsonian gauge couplings must transform according to eqs. (3.9). However, only for the factorizable orbifolds *all* of the $SL(2,\mathbf{Z})$ transformations (3.4) are true symmetries of the string theory; in the non-factorizable case, the group of the true modular symmetries is only a subgroup of the $[SL(2,\mathbf{Z})]^n$. Clearly, only the true symmetries constrain the moduli dependence of the Wilsonian gauge couplings, which therefore need not be exactly as in eqs. (3.16).

As an example, let us consider the $[SU(3)\times SO(8)]/Z_6$ orbifold where the period lattice of the internal six-torus is a deformation of the $SU(3)\times SO(8)$ root lattice and the orbifold group $D=Z_6$ is generated by the $\Theta=(e^{2\pi i/3}, e^{2\pi i/6}, -1)$. This orbifold has the usual three T^i moduli for each of the three eigenplanes of Θ plus the U^3 modulus for the third complex plane. However, the modular group for the third plane is not the full $[SL(2,\mathbf{Z})]^2$ but only its proper subgroup $[\Gamma^0(3)]^2$, *i.e.*, the integer b in eq. (3.4) for either T^3 or U^3 must be divisible by 3. Unlike the full $SL(2,\mathbf{Z})$ group, which has no holomorphic invariants without either singularities or unacceptable rate of growth in the decompactification limit, the $\Gamma^0(3)$ group does have several invariants of this kind, namely

$$\log\eta\left(\frac{iM+\lambda}{3}\right) - \log\eta(iM) \qquad \text{for } \lambda=0,1,2. \tag{3.23}$$

Consequently, for this orbifold, the field theoretical constraints specify the moduli dependence of the gauge couplings only up to an arbitrary linear combination of the invariants (3.23). The coefficients of such invariants apparently can only be determined by the string theory at the one-loop level. The techniques for calculating the string-theoretical threshold corrections for the non-factorizable orbifolds were developed by Mayr and Stieberger[18]; their explicit results show that the $\Delta_a(M)$ indeed include holomorphic invariants such as (3.23).

4. $(2,2)$ Supersymmetric Vacua.

In the previous section we saw how analytic knowledge of the moduli dependence of the orbifolds' tree-level couplings can be used to deduce (or at least severely constrain) the one-loop corrections to the gauge couplings. Now we turn our attention to the $(2,2)$-supersymmetric vacua of the heterotic string, for which we also have some analytic knowledge of the moduli-dependent tree-level couplings.[34,35,37,50−55] Calabi-Yau compactifications of the ten-dimensional heterotic string are the best-known examples of such vacua. However, the $(2,2)$ vacua can be defined and studied in string-theoretical terms without any reference to the geometry of the six compact dimensions and without even assuming that the internal SCFT has any geometrical interpretation at all. Let us therefore begin this section with a brief review of the *generic* $(2,2)$ vacua and their known properties.

From the world-sheet point of view, a $(2,2)$ vacuum is defined by the following two features: First, the internal $c=(22,9)$ SCFT contains an $SO(10)\times E_8$ left-moving Kac-Moody algebra ($k=1$ for both the $SO(10)$ and the E_8 factors).[⋆] Second, the remaining $c=(9,9)$ part of the SCFT has $N=(2,2)$ world-sheet supersymmetry and both the left- and the right-moving $N=2$ superalgebras have quantized $U(1)$ charges F and $\overline{F}$ (these charges are always equal to the respective fermion numbers, hence the notation). As usual, the right-moving $N=2$ superalgebra is responsible for the $N=1$ spacetime supersymmetry; it is the left-moving $N=2$ superalgebra that leads to the peculiar features of the $(2,2)$ vacuum families.

⋆ Alternatively, the internal SCFT of a $(2,2)$ vacuum may contain an $SO(26)$ Kac-Moody algebra (also at level $k=1$) instead of an $SO(10)\times E_8$. In this article, however, we focus on the $SO(10)\times E_8$ case.

The gauge group of any $(2,2)$ vacuum family is always $G=E_6\times E_8$[†] and the matter fields Q^I consist of $h_{(1,1)}$ **27** multiplets of the E_6, $h_{(1,2)}$ $\overline{\mathbf{27}}$ multiplets[‡] and some gauge singlets; none of the light matter fields is charged under the E_8 group. We limit our discussion to the moduli that preserve the $(2,2)$ nature of the vacuum. Such moduli are in one-to-one correspondence with the charged matter fields and thus we distinguish between the $h_{(1,1)}$ moduli T^i related to the **27** matter fields and the $h_{(1,2)}$ moduli U^i related to the $\overline{\mathbf{27}}$'s. At the tree level, these two types of moduli form separate moduli spaces and the Kähler function is a sum

$$\hat{K} \;=\; \hat{K}_1(T,\overline{T}) \;+\; \hat{K}_2(U,\overline{U}). \tag{4.1}$$

Furthermore, both moduli spaces have special Kähler geometries, so both $\hat{K}_1$ and $\hat{K}_2$ can be written in terms of holomorphic pre-potentials $\mathcal{F}_1(T)$ and $\mathcal{F}_2(U)$.[56,34,35] Unfortunately, loop corrections do not respect the special Kähler geometry of the moduli space, so the holomorphicity of the prepotentials does not lead to any non-renormalization theorems for the Kähler functions.

The one-to-one correspondence between the moduli and the charged matter fields results in a close relation between their respective metrics: In matrix

† For some special vacua within the family (corresponding to special points or subspaces of the moduli space), additional vector bosons and matter fields might become massless. In principle, the low-energy physics of such special vacua (and their close neighbors) should be described by a different EQFT that accounts for the additional light particles and it may also re-interpret some of the moduli as combinations of the matter fields. Such re-analysis is absolutely essential for studying the "accidental" enlargements of the gauge group and their couplings. On the other hand, provided the "accidental" gauge symmetries commute with the $E_6\times E_8$ and all the accidentally light matter fields are neutral under the $E_6\times E_8$ (both conditions are true in all the known examples), none of these extra fields have any one-loop-level impact on the gauge couplings of E_6 and E_8. Therefore, in this section, we limit our attention to the E_6 and E_8 gauge couplings for generic $(2,2)$ vacua.

‡ For Calabi-Yau compactifications, $h_{(1,1)}$ and $h_{(1,2)}$ are the Hodge numbers of the manifold, hence the notation. For generic $(2,2)$ vacua, $h_{(1,1)}$ and $h_{(1,2)}$ are simply integer parameters.

notations,

$$Z^{(\mathbf{27})} \ = \ G_1 \times \exp \tfrac{1}{3}\big(\hat{K}_2 - \hat{K}_1\big) \qquad \text{and} \quad Z^{(\overline{\mathbf{27}})} \ = \ G_2 \times \exp \tfrac{1}{3}\big(\hat{K}_1 - \hat{K}_2\big) \quad (4.2)$$

where G_1 and G_2 are the moduli metrics

$$(G_1)_{\bar{\imath}j} \ = \ \frac{\partial^2 \hat{K}_1}{\partial \overline{T}^i \partial T^j} \qquad \text{and} \quad (G_2)_{\bar{\imath}j} \ = \ \frac{\partial^2 \hat{K}_2}{\partial \overline{U}^i \partial U^j}\,. \qquad (4.3)$$

Eqs. (4.2) and (4.1) are valid only at the tree level of the heterotic string, but that is all we need to determine the non-harmonic parts of the one-loop-level threshold corrections to the gauge couplings. Indeed, substituting eqs. (4.2) into eq. (2.10) and taking into account the group-theoretical factors,⋆ we obtain[57,17]

$$\begin{aligned} \tilde{\Delta}_{E_8} \ &= \ \mathrm{Re}\, f_8^{(1)} \ - \ 30\hat{K}_1 \ - \ 30\hat{K}_2, \\ \tilde{\Delta}_{E_6} \ &= \ \mathrm{Re}\, f_6^{(1)} \ + \ (5h_{(1,1)} + h_{(1,2)} - 12)\hat{K}_1 \ - \ 6\log\det G_1 \\ &\qquad + \ (5h_{(1,2)} + h_{(1,1)} - 12)\hat{K}_2 \ - \ 6\log\det G_2\,. \end{aligned} \qquad (4.4)$$

Of particular interest is the difference between these two equations,

$$\begin{aligned} \tilde{\Delta}_{E_6} \ - \ \tilde{\Delta}_{E_8} \ = \ \mathrm{Re}\, f_{6-8}^{(1)} \ &+ \ 6(3 + h_{(1,1)} - \tfrac{1}{12}\chi)\hat{K}_1 \ - \ 6\log\det G_1 \\ &+ \ 6(3 + h_{(1,2)} + \tfrac{1}{12}\chi)\hat{K}_2 \ - \ 6\log\det G_2 \end{aligned} \qquad (4.5)$$

($\chi = 2[h_{(1,1)} - h_{(1,2)}]$ is the Euler number and $f_{6-8}^{(1)} \equiv f_6^{(1)} - f_8^{(1)}$). The non-harmonic part on the right hand side here is precisely 12 times the "holomorphic anomaly" of the topological index F_1 of Bershadsky, Cecotti, Ooguri and Vafa.[36] Furthermore, using classical six-dimensional geometry, they proved that for the large-radius Calabi-Yau threefolds and their mirror images one indeed has

$$\tilde{\Delta}_{E_6} \ - \ \tilde{\Delta}_{E_8} \ = \ 12F_1 \qquad (4.6)$$

and hence F^{top} — the holomorphic part of the F_1 — is the same as $\frac{1}{12}f_{6-8}$. We will show momentarily that eq. (4.6) actually holds for all the $(2,2)$ vacua

⋆ In the notations of eq. (2.9), $T(E_8) = 30$, $T(E_6) = 12$, $T_{E_6}(\mathbf{27}) = T_{E_6}(\overline{\mathbf{27}}) = 3$; thus $\mathbf{c}_{E_8} = -30$ and $\mathbf{c}_{E_6} = 3h_{(1,1)} + 3h_{(1,2)} - 12$.

of the heterotic string, regardless of whether such a vacuum has a geometrical interpretation of any kind.

Eqs. (4.4) are field-theoretical constraints based upon the tree-level properties of the $(2,2)$ vacua. However, these vacua also have characteristic features that become important at the one-loop level of the string theory. In particular, the fact that the internal $c = (22,9)$ world-sheet SCFT splits into an $c = (9,9)$ SCFT plus the $SO(10) \times E_8$ $(k = 1)$ Kač-Moody algebra allows us to factorize the trace in eq. (2.17):

$$\begin{aligned} \mathcal{B}_a \ = \ \frac{-1}{2\eta^2(\tau)} \sum_{\text{even }\mathbf{s}} \mathrm{Tr}_{(s_1,R)} \Big((-1)^{s_2 F} \, (-1)^{\overline{F} - \frac{3}{2}\overline{F}} \, q^{L - \frac{3}{8}} \bar{q}^{\bar{L} - \frac{3}{8}} \Big)_{(9,9)} \\ \times \, \mathrm{Tr}_{s_1} \Big((-1)^{s_2 F} \Big(T_{(a)}^2 - \tfrac{1}{8\pi\tau_2} \Big) \, q^{L - \frac{13}{24}} \Big)_{SO(10)\times E_8} \end{aligned} \qquad (4.7)$$

where the summation over the NSR boundary conditions now refers to the left-moving world-sheet fermions. The second trace in this formula distinguishes between the gauge couplings of the E_8 and of the E_6 (for the E_6 one uses a generator $T_{(a)}$ in $SO(10) \subset E_6$ subgroup), but it is totally insensitive to specific properties of a particular $(2,2)$ vacuum. All such traces (altogether six, for the two gauge groups and the three even NSR boundary conditions) can be easily obtained from the characters of the $SO(10) \times E_8$ Kač-Moody algebra or even from the partition functions $Z_{E_8}(\tau)$ and $Z_{SO(10)}(\mathbf{s},\tau) = Z_\Psi^5(\mathbf{s},\tau)$ (Z_Ψ is the partition function of one complex free fermion or two real ones); in terms of the partition functions, the second line of eq. (4.7) equals to

$$Z_{E_8}(\tau)\, Z_\Psi^5(\mathbf{s},\tau) \times \frac{1}{2\pi i}\frac{\partial}{\partial\tau} \begin{cases} \big(\log Z_\Psi(\mathbf{s},\tau) \ + \ \frac{1}{2}\log\big(\tau_2|\eta(\tau)|^4\big)\big) & \text{for } E_6, \\ \big(\frac{1}{8}\log Z_{E_8}(\tau) \ + \ \frac{1}{2}\log\big(\tau_2|\eta(\tau)|^4\big)\big) & \text{for } E_8. \end{cases} \qquad (4.8)$$

We do not see how one can further simplify eqs. (4.8), but we can simplify

the difference of the two expressions. Using the partition function identities

$$\begin{aligned} Z_\Psi(NS,NS)\times Z_\Psi(NS,R)\times Z_\Psi(R,NS) &= 2,\\ Z_\Psi^4(NS,NS)-Z_\Psi^4(NS,R)-Z_\Psi^4(R,NS) &= 0,\\ Z_\Psi^8(NS,NS)+Z_\Psi^8(NS,R)+Z_\Psi^8(R,NS) &= Z_{E_8}, \end{aligned}\tag{4.9}$$

it is easy to show that for each of the three even NSR boundary conditions,

$$Z_{E_8}(\tau)\,Z_\Psi^5(\mathbf{s},\tau)\times\frac{\partial}{\partial\tau}\left(\log Z_\Psi(\mathbf{s},\tau)\;-\;\tfrac{1}{8}\log Z_{E_8}(\tau)\right)\;=\;-24(-1)^{s_1+s_2}\,\frac{\partial Z_\Psi(\mathbf{s},\tau)}{\partial\tau}\,.\tag{4.10}$$

The left hand side of this equation involves partition functions of altogether 26 real fermions (10 for the $SO(10)$ and 16 for the E_8), but only two real fermions appear on the right hand side. Such drastic reduction in fermionic degrees of freedom is characteristic of the so-called *bosonic/supersymmetric* map between the heterotic string and the type II superstring.[58] In the context of (4.8) and (4.7), the bosonic/supersymmetric map (4.10) immediately gives us

$$\begin{aligned}\mathcal{B}_6\;-\;\mathcal{B}_8\;=\;&\frac{12}{\eta^2(\tau)}\sum_{\text{even }\mathbf{s}}(-1)^{s_1+s_2}\,\frac{\partial Z_\Psi(\mathbf{s},\tau)}{2\pi i\partial\tau}\\ &\times\,\mathrm{Tr}_{(s_1,R)}\left((-1)^{s_2F}\,(-1)^{\overline{F}-\frac{3}{2}}\overline{F}\,q^{L-\frac{3}{8}}\bar{q}^{\bar{L}-\frac{3}{8}}\right)_{(9,9)}.\end{aligned}\tag{4.11}$$

Now consider the left-moving $N=2$ world-sheet supersymmetry of the $c=(9,9)$ SCFT. The left-moving and the right-moving $N=2$ superalgebras of the $(2,2)$ vacua are complex conjugates of each other and satisfy exactly the same F charge quantization condition; consequently, both superalgebras give rise to the same kind of Riemann identities between the NSR sectors of the $c=(9,9)$ SCFT. In Appendix A, we show that the right-moving $N=2$ superalgebra reduces the sum over three even NSR sectors in eq. (2.13) to a single Ramond-Ramond trace in eq. (2.17). In that proof, only the right-moving degrees of freedom play any role while the left-moving degrees of freedom simply come along for the ride, but

of course, exactly the same identity would also apply to the left-moving side of an $(2,2)$ supersymmetric SCFT. This is precisely the situation we have in eq. (4.11), whose left-moving side looks exactly like the right-moving side of eq. (2.13) and thus can be reduced in exactly the same way. The result is

$$\mathcal{B}_6\;-\mathcal{B}_8\;=\;6\;\mathrm{Tr}_{R,R}\left((-1)^{F-\overline{F}}\,F\overline{F}\,q^{L-\frac{3}{8}}\bar{q}^{\bar{L}-\frac{3}{8}}\right)_{(9,9)}\tag{4.12}$$

where the boundary conditions are Ramond-Ramond on both sides of the $c=(9,9)$ SCFT and no other world-sheet degrees of freedom are involved.

In the path-integral formulation of $N=(2,2)$ SCFT, the totally-Ramond characters (Ramond-Ramond for both sides of the world-sheet) are given by the zero modes of the conformal fields⋆ and thus behave as generalized supersymmetry indices of the theory. The particular index (4.12) and its $d^2\tau$ integral

$$F_1\equiv\tfrac{1}{2}\int\frac{d^2\tau}{\tau_2}\,\mathrm{Tr}'_{R,R}\left((-1)^{F-\overline{F}}\,F\overline{F}\,q^{L-\frac{3}{8}}\bar{q}^{\bar{L}-\frac{3}{8}}\right)_{(9,9)}\tag{4.13}$$

was first encountered in ref. 59 and later studied in more detail in ref. 36. Comparing this F_1 with eqs. (4.12), (2.12) and (1.5), we immedaitely see $12F_1=\Delta_{E_6}-\Delta_{E_8}=\tilde{\Delta}_{E_6}-\tilde{\Delta}_{E_8}$; this concludes our proof of eq. (4.6).

Thus far we discussed the general features of all the $(2,2)$ vacua, regardless of their geometrical interpretation or lack thereof. Let us now turn to the vacua which are related to the Calabi-Yau threefolds and consider how the size of the internal threefold affects the four-dimensional gauge couplings. The "overall radius" R of the threefold is one of its $(1,1)$ moduli; according to eq. (4.1), it does not affect the $(1,2)$ moduli U^i, so we may safely disregard the latter in the following discussion. In terms of the cohomologically defined $(1,1)$ moduli T^i, all

⋆ The non-zero modes come in supermultiplets — the totally-Ramond boundary conditions preserve all of the world-sheet supersymmetries — and the bosonic non-zero modes cancel against the fermionic ones and vice verse.

the $\operatorname{Re} T^i$ are proportional to R^2 while the torsions $\operatorname{Im} T^i$ are radius-independent. For each torsion, there is a discrete Peccei-Quinn symmetry; these symmetries involve neither Kähler transformations nor rescalings of the charged matter fields and hence should leave the Wilsonian gauge couplings $\operatorname{Re} f_a$ invariant. Since the f_a are holomorphic functions of T^i, this immediately implies

$$f_a^{(1)}(T) \ = \sum_i \omega_{a,i} T^i \ + \ \text{const} \ + \ \delta f_a\left(\exp(-2\pi T)\right), \tag{4.14}$$

where $\omega_{a,i}$ are some rational proportionality constants and the last term is exponentially small in the large radius limit. Thus, in that limit we have

$$\operatorname{Re} f_a^{(1)} \ \to \omega_a \ R^2 + \text{const}, \tag{4.15}$$

and the large radius behavior of the $f_a^{(1)}$ depends on whether the constant ω_a vanishes or not, and this cannot be determined by the Peccei-Quinn symmetries alone.

Before we turn to string-theoretical reasons determining ω_a, let us consider the field-theoretical non-harmonic contributions to the threshold corrections $\tilde{\Delta}_a$. For Calabi-Yau manifolds that are both large and smooth, *i.e.*, when *all of the* $\operatorname{Re} T^i$ *are large*, the Kähler function $\hat{K}_1$ can be approximated as[60,35]

$$\hat{K}_1(T,\overline{T}) \ \approx \ -\log\Bigl(d_{ijk}(\operatorname{Re} T^i)(\operatorname{Re} T^j)(\operatorname{Re} T^k)\Bigr), \tag{4.16}$$

and hence $\exp(\hat{K}_1)$ is proportional to R^{-6} while the moduli metric matrix G_1 scales like R^{-4}. Substituting these scaling laws into eqs. (4.4), collecting all the $\log R^2$ terms and using eq. (4.15) gives us[61]

$$\tilde{\Delta}_a \ \approx \ \omega_a \ R^2 - \mathbf{b}_a \log R^2 \ + \ \text{const} \tag{4.17}$$

for both the E_6 and the E_8 couplings. We emphasize that this result depends on all of the $\operatorname{Re} T^i$ being large and does not apply to degenerate manifolds for which some of the T^i are frozen at zero values; indeed, the orbifold results of section 3 generally disagree with eq. (4.17).†

The curious coincidence between the coefficients of the $\log R^2$ term in eqs. (4.17) and the β-function coefficients for the respective gauge couplings suggests that perhaps in eq. (4.15), $\omega_a = 0$ and the entire radius-dependence of the threshold corrections $\tilde{\Delta}_{E_6}$ and $\tilde{\Delta}_{E_8}$ amounts to changing the effective threshold scale from the string scale $M_{\text{string}} \sim 1/\sqrt{\alpha'}$ to the Kaluza-Klein scale $1/R$.[49] However, eq. (4.6) can be used to show that the leading term in the large-radius limit of $\Delta_{E_6} - \Delta_{E_8}$ is proportional to the R^2 rather than to the $\log R^2$ and hence $\omega_6 - \omega_8 \neq 0$. Indeed, ref. 36 gives the large-radius limit of the topological index F_1 as

$$F_1 \to \frac{1}{6} \sum_i \ \operatorname{Re} T^i \int_{\mathcal{M}} \mathbf{k}_i \wedge \mathbf{C}_2 \ = \ \frac{1}{96\pi^2} \int_{\mathcal{M}} \|\mathcal{R}\|^2 \tag{4.18}$$

where $\mathbf{C}_2$ is the second Chern class of the Calabi-Yau threefold $\mathcal{M}$, $\mathcal{R}$ is its Riemannian curvature tensor and the $\mathbf{k}_i$ form a basis of the cohomology group $H^{(1,1)}$. The left hand side here is obviously proportional to the R^2 while the right hand side is positive definite; together, they guarantee that F_1 and thus the difference $\tilde{\Delta}_{E_6} - \tilde{\Delta}_{E_8}$ indeed grows like R^2 in the large-radius limit.

In terms of the Wilsonian couplings $f_6^{(1)}$ and $f_8^{(1)}$, eq. (4.18) tells us that

$$f_6^{(1)}(T) \ - \ f_8^{(1)}(T) \ = \ 2 \sum_i \ T^i \int_{\mathcal{M}} \mathbf{k}_i \wedge \mathbf{C}_2 \ + \ \text{const} \ + \ \delta f_{6-8}\left(\exp(-2\pi T)\right). \tag{4.19}$$

Unfortunately, we do not have a second equation of this kind that would determine separate $R \to \infty$ limits of the $f_6^{(1)}$ and of the $f_8^{(1)}$. In general, all we can

† For large, smooth manifolds, the entire moduli metric matrix G_1 is proportional to R^{-4} while the entire Z matrix for the $\mathbf{27}$ matter fields is proportional to R^{-2} (*cf.* eq. (4.2)). For singular $(2,2)$ orbifolds, the same is true for the *untwisted* moduli and $\mathbf{27}$ matter fields, but the *twisted* fields have quite different scaling properties. For example, for the Z_3 orbifold, the Z matrix for the twisted $\mathbf{27}$'s is proportional to R^{-4} instead of the usual R^{-2}. Of course, once the sharp points of an orbifold are blown up (and the blow-up radii increase proportionately to the overall radius R), we do recover the usual R^{-2} scaling properties of the $Z^{\mathbf{27}}_{\text{twisted}}$ matrix and thus restore the validity of eqs. (4.17).

say is that they have the general form (4.14) and do not grow faster than R^2. However, for some specific Calabi-Yau threefolds, the entire analytic form of both $f_6^{(1)}$ and $f_8^{(1)}$ can be deduced by essentially the same techniques as we used in section 3.

As an example, consider the quintic threefold analyzed by Candelas, de la Ossa, Green and Parkes.[37] It has $h_{(1,2)} = 101$ but $h_{(1,1)} = 1$, so its $(1,1)$ moduli space needs only one complex coordinate; Candelas *et al.* found it convenient to work with the "mirror coordinate" ψ instead of the flat coordinate T. According to eqs. (4.2), such coordinate transformation also entails a linear redefinition of the charged matter fields, which means that one should also use different sets of Wilsonian gauge couplings f_a in different coordinate pictures (*cf.* eq. (3.3)). We should also account for a possible Kähler transformation between different coordinate pictures, but fortunately, this transformation is trivial for the two particular pictures of the quintic discussed here. Thus,

$$f_8^{(1)}(\psi) \;=\; f_8^{(1)}(T) \qquad \text{but} \quad f_6^{(1)}(\psi) \;=\; f_6^{(1)}(T) \;-\; 12\frac{d\psi}{dT}\,. \tag{4.20}$$

In terms of ψ, some of the modular transformations are monodromies that map ψ onto itself while others map $\psi \to e^{2\pi i/5}\psi$; thus all physical quantities must be single-valued functions of the ψ^5. The Kähler function $\hat{K}_1(\psi,\bar{\psi})$ determined in ref. 37 is invariant under all the modular transformations, but the Yukawa coupling $Y_{\mathbf{27}}$ of the only $\mathbf{27}$ multiplet of the theory transforms like

$$Y_{\mathbf{27}}(\psi) \;=\; \frac{(2\pi i)^3}{25}\,\frac{\psi^2}{1-\psi^5} \;\to\; e^{4\pi i/5}\,Y_{\mathbf{27}} \quad \text{when} \quad \psi \to e^{2\pi i/5}\psi, \tag{4.21}$$

which implies that the ψ-changing modular transformations are R-symmetries of the charged fields:

$$Q_{\mathbf{27}} \;\to\; e^{-2\pi i/15}\,Q_{\mathbf{27}}\,, \quad Q^i_{\overline{\mathbf{27}}} \;\to\; e^{+2\pi i/15}\,Q^i_{\overline{\mathbf{27}}} \quad \text{and} \quad W \;\to\; e^{+2\pi i/5}\,W \tag{4.22}$$

when $\psi \to e^{2\pi i/5}\psi$. Assuming as usual that the dilaton superfield S is inert under all modular transformations, we apply eqs. (3.3) to the transformations

(4.22) and conclude that both of the Wilsonian threshold corrections $f_8^{(1)}$ and $f_6^{(1)}$ must be single-valued (modulo $4\pi i$) functions of ψ, and furthermore,

$$\left.\begin{aligned} f_8^{(1)}(e^{2\pi i/5}\psi) \;&=\; f_8^{(1)}(\psi) \\ f_6^{(1)}(e^{2\pi i/5}\psi) \;&=\; f_6^{(1)}(\psi) \;-\; \tfrac{4\pi i}{5} \end{aligned}\right\}\ \text{modulo } 4\pi i. \tag{4.23}$$

The $(2,2)$ vacuum family of the quintic threefold includes the Gepner $[3]^5$ model[62]; that particular vacuum corresponds to $\psi = 0$. The Gepner model has four massless abelian gauge fields as well as four massless matter superfields that are not present in the spectra of the generic vacua in the same family; however, all these "accidentally" massless fields are neutral under the $E_8 \times E_6$ gauge group. Therefore, the physical gauge couplings g_{E_8} and g_{E_6} should have no singularities at the "Gepner point" $\psi = 0$. At the same point, the metric $G_{\bar{\psi}\psi}$ of the modulus ψ is non-singular, but the Kähler function has a logarithmic singularity, $\hat{K}_1(\bar{\psi},\psi) = -\log|\psi|^2 + \text{finite}$. According to eqs. (4.4), this singularity should be canceled by appropriately singular terms in the Wilsonian corrections $f_8^{(1)}$ and $f_6^{(1)}$. Thus, for $\psi \to 0$,

$$\begin{aligned} f_8^{(1)}(\psi) \;&=\; \;\;-60\log\psi \;+\; \text{finite}, \\ f_6^{(1)}(\psi) \;&=\; +188\log\psi \;+\; \text{finite}; \end{aligned} \tag{4.24}$$

note that the modular transformations of the logarithmic terms here agrees with eqs. (4.23).

Besides the spurious Kähler singularity at the Gepner point $\psi = 0$, the $(1,1)$ moduli space of the quintic has two genuine, physical singularities: $\psi \to \infty$ is the large radius limit of the threefold, and at $\psi^5 = 1$, the mirror threefold suffers from conifold degeneration.[37] In the large radius limit one has $T \approx \log\psi^5$; thus, in light of eqs. (4.14) and (4.20), the Wilsonian gauge couplings have at most logarithmic divergences as $\psi \to \infty$. Taking also into account the modular

transformation properties (4.23) of the two $f_a^{(1)}$, their $\psi \to 0$ limits (4.24) and the requirement that there should be no singularities of any kind except at $\psi = 0$, $\psi = \infty$ or $\psi^5 = 1$ we arrive at

$$\begin{aligned} f_8^{(1)}(\psi) &= -12 \log \psi^5 \;+\; (\omega_8 + 12) \log(\psi^5 - 1) \;+\; p_8(\psi^5 - 1), \\ f_6^{(1)}(\psi) &= +\tfrac{188}{5} \log \psi^5 \;+\; (\omega_6 - 40) \log(\psi^5 - 1) \;+\; p_6(\psi^5 - 1), \end{aligned} \tag{4.25}$$

where the coefficients ω_8 and ω_6 are exactly as in eqs. (4.14). The functions p_8 and p_6 here must be single-valued (in terms of ψ^5) and non-singular anywhere except at $\psi^5 = 1$; such functions may have poles or essential singularities at that point, but no logarithmic or other singularities that require branch cuts.

Now consider the physics of the conifold limit $\psi^5 \to 1$. In that limit, the Kähler function $\hat{K}_1$ is finite while the metric $G_{\bar{\psi}\psi}$ has only a mild logarithmic singularity; the leading divergences of the threshold corrections $\tilde{\Delta}_{E_8}$ and $\tilde{\Delta}_{E_6}$ should therefore come from the Wilsonian terms in eqs. (4.4). In light of eqs. (4.25) one might ask: *How can a gauge coupling have a pole or an essential singularity at $\psi^5 = 1$?* Generally, threshold corrections to gauge couplings become singular when otherwise massive charged fields become massless for some particular values of the moduli, but such divergences are always logarithmic with respect to the "accidentally" vanishing masses. Thus, there are only two ways to get a pole or any other singularity that is stronger than logarithmic: The first way is for the masses to vanish not like powers of $(\psi^5 - 1)$ but exponentially or faster; this is rather implausible in terms of the known geometry of the conifold limit. The second way is to have an infinite number of charged fields that all become massless at the same time, which means that in string-theory, the conifold limit would be equivalent to some kind of a decompactification and at $\psi^5 = 1$ we would effectively have five or more non-compact spacetime dimensions. In this scenario, the rate at which $f_a^{(1)}$ grow when $\psi^5 \to 1$ can be limited in essentially the same way as we have limited the large-radius growth of the orbifolds' f_a in the previous section. Let us skip the technical details of this argument; the result

is that if the conifold were to decompactify, the $f_{6,8}^{(1)}$ could be no more singular than $1/(\psi^5 - 1)^2$. However, the geometry of the conifold does not seem to support infinitesimally short strings wrapping around non-contractible loops and we do not see what other string modes could lead to an effective decompactification of the conifold limit. Let us therefore *conjecture* that there is no decompactification and only a finite number of particles become massless at $\psi^5 = 1$. In this case, the threshold corrections cannot have any poles or essential singularities at $\psi^5 = 1$; in terms of eqs. (4.25), this means $p_8 = p_6 =$ const. (Of course, a "constant" part of an $f_a^{(1)}(\psi)$ is actually a function of the 101 moduli U^i, but the analytic form of *that* function is beyond the scope of the present discussion.)

The question of the logarithmic singularities of eqs. (4.25) at $\psi^5 = 1$ is more subtle since such singularities require only a finite number of otherwise massive charged fields to become massless. We believe however that even this does not happen for the E_8 coupling. Indeed, consider the heterotic string vertices of hypothetical massless particles with non-trivial E_8 charges. The $k = 1$ Kač-Moody algebra responsible for the E_8 has no sources of charge other that the Kač-Moody currents $J_{(a)}(z)$ themselves. Therefore, a heterotic vertex of any massless E_8-charged boson (because of the spacetime supersymmetry, it is enough to consider the bosons) has to factorize into a product of the form $e^{ip\cdot X} J_{(a)} \left(\Phi + \frac{i}{2}(p \cdot \psi)\Psi\right)$ where the operators Φ and Ψ have conformal dimensions $h = (0, 1)$ and $h = (0, \frac{1}{2})$, respectively. These dimensions mean that $\Phi(\bar{z})$ is a right-moving current while $\Psi(\bar{z})$ is a right-moving free fermion; the ordinary gauge bosons of the E_8 come from such operators in the spacetime part of the world-sheet SCFT, $\Phi = \partial X^\mu$ and $\Psi = \psi^\mu$. However, were there additional operators of this kind in the internal part of the SCFT, the free right-moving fermion Ψ would have a zero mode in the Ramond sector. That zero mode would be inseparable from the zero modes of the four ψ^μ and thus would allow changing the spacetime chirality of any fermionic particle without changing the rest of its quantum numbers; in other words, there would be absolutely no chirality in the particle spectrum of the four-dimensional

theory.[⋆]

Although the conifold limit of the quintic threefold corresponds to a somewhat singular $(2,2)$ vacuum, we do not believe it is singular enough to eliminate the non-zero Euler number of the theory and completely remove the chirality of its spacetime fermions. Therefore, we find it implausible that any particle that becomes accidentally massless in the conifold limit can carry an E_8 charge. The E_8 gauge coupling thus cannot have even a logarithmic singularity at the conifold point $\psi^5 = 1$ and the first eq. (4.25) reduces to simply

$$f_8^{(1)}(\psi) \;=\; -12\log\psi^5 \;+\; \text{const.} \tag{4.26}$$

Among other things, this formula gives us the exact large-radius limit of the E_8 coupling: in terms of eq. (4.15), $\omega_8 = -12$.

For the gauge coupling of the E_6, the situation is somewhat different. The same argument we have just used for the E_8 also rules out any accidentally massless particles in the adjoint representation of the E_6. Furthermore, the non-trivial chirality of the conifold limit also rules out any accidental enlargement of the E_6 gauge group to an E_7 or an E_8. What we cannot rule out, and what we believe might indeed happen is the accidental masslessness of an $\mathbf{27}+\overline{\mathbf{27}}$ matter multiplet. As a result, the E_6 coupling would diverge logarithmically, and while we cannot calculate the coefficient of such divergence without knowing exactly how many $\mathbf{27}+\overline{\mathbf{27}}$ multiplets do become massless and the way their masses depend upon ψ^5-1, we can be sure of its sign. In terms of eq. (4.25), we must have $(\omega_6 - 40) \leq 0$.

At this point, we again turn to the results of ref. 36 who have calculated the topological integral in eq. (4.19) for the quintic and thus determined the large-

⋆ This argument, adapted from ref. 63, applies to any four-dimensional vacuum of the heterotic string, spacetime supersymmetric or otherwise. For any *level* $k=1$ Kač-Moody algebra, if there are any massless scalars in the adjoint representation of the gauge group, then the spacetime fermions cannot have any chirality.

radius limit (and hence the entire analytic form) of the difference $f_{6-8}(\psi)$. In our notations, their result amounts to $\omega_6 - \omega_8 = 50$, which indeed agrees with $\omega_8 = -12$ and $\omega_6 \leq 40$. Thus we now know the exact analytic form of the Wilsonian E_6 coupling,

$$f_6^{(1)}(\psi) \;=\; +\tfrac{188}{5}\log\psi^5 \;-\; 2\log(\psi^5-1) \;+\; \text{const.} \tag{4.27}$$

We conclude this article by extending the above analysis of the quintic threefold to the three other threefolds that also have $h_{(1,1)} = 1$ and similar singularities of the $(1,1)$ moduli space.[51] (Analysis of Calabi-Yau threefolds with $h_{(1,1)} \geq 2$ is similar in principle but technically more difficult because of more complicated singularities.) Following the notation of refs. 51, the modular-invariant coordinate of the $(1,1)$ moduli space is ψ^k where $k = 6$, 8 or 10, depending on a particular model, and the singularities are at $\psi \to \infty$ (the large radius limit) and $\psi^k = 1$ (a conifold singularity); there is also a spurious singularity at the Gepner point $\psi = 0$. In this notation the quintic threefold — which follows the same pattern — corresponds to $k = 5$. The asymptotic behavior of the $\hat{K}_1$ and $G_{\bar\psi\psi}$ is similar for all these models; it is spelled out in detail in refs. 51. Given these data — and the topological integrals (4.19) computed in ref. 36, — we obtain

$$\begin{aligned} f_8^{(1)}(\psi) \;&=\; -60\,\log\psi \;+\; \text{const.},\\ f_6^{(1)}(\psi) \;&=\; A(k)\,\log\psi \;-\; 2\log(\psi^k-1) \;+\; \text{const.}, \end{aligned} \tag{4.28}$$

where $A(5) = 188$ (*cf.* eq. (4.27)), $A(6) = 192$, $A(8) = 296$ and $A(10) = 288$.

Remarkably, all four models have exactly the same logarithmic divergence of the E_6 coupling in the conifold limit $\psi^k \to 1$. The coefficient (-2) of this divergence tells us that some $\mathbf{27}+\overline{\mathbf{27}}$ multiplets do become "accidentally" light in the conifold limit and that the product of their masses behaves like $\left(\psi^k-1\right)^{1/6}$. It would be interesting to verify this result by a direct string calculation of the masses.

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APPENDIX A
Riemann Identities for Threshold Corrections.

In terms of the internal $c = (22, 9)$ world-sheet SCFT, unbroken $N = 1$ supersymmetry of the four-dimensional spacetime requires extended $N = (0, 2)$ world-sheet supersymmetry (rather than just $N = (0, 1)$ required by the heterotic string itself). The current algebra of this extended supersymmetry contains an abelian current $J(\bar{z})$ whose charge $\oint J$ should be quantized. Together, J and its quantized charge describe a free chiral boson ($J = \sqrt{3}\partial H$) of radius $\sqrt{3}$, which is a universal $c = (0, 1)$ part of the internal SCFT of any spacetime-supersymmetric vacuum.[64] Among other things, this universal part is responsible for the NSR sectors of the internal SCFT; the remaining $c = (22, 8)$ part — the part that differs from vacuum to vacuum — is the same in all the 4^{genus} NSR sectors. Joining the $c = (22, 8)$ and the $c = (0, 1)$ SCFTs together involves 3 conjugacy classes and hence only 3^{genus} sectors. This fact leads to linear relations between the partition functions and characters of the combined world-sheet SCFT for different NSR sectors; such relations are generally known as *Riemann identities*.[65,58]

The quintessential Riemann identities of the $N = 1$, $d = 4$ spacetime supersymmetry are identities for the characters of the E_6/D_4 coset algebra, which combines the internal H boson with the bosonized fermionic superpartners of the two transverse space coordinates. All the other Riemann identities can be derived from these and at genus = 1, there is one such identity, which reads[65]

$$\sum_{\mathbf{s}}(-1)^{s_1+s_2}\,\mathrm{Ch}_\rho^{E_6/D_4}(\mathbf{s},\nu_\Psi,\nu_H,\tau) = 2\,\mathrm{Ch}_\rho^{E_6/D_4}(RR,\tfrac{1}{2}\nu_\Psi-\tfrac{\sqrt{3}}{2}\nu_H,\tfrac{1}{2}\nu_H+\tfrac{\sqrt{3}}{2}\nu_\psi,\tau). \tag{A.1}$$

Here the subscript ρ labels the three conjugacy classes of the E_6, RR stands for the Ramond-Ramond sector $\mathbf{s} = (1, 1)$ and the linear transformation of the (ν_Ψ, ν_H) 2-vector on the right hand side is simply a $\pi/3$ rotation. The one-loop characters Ch^{E_6/D_4} can be expressed in Hamiltonian terms according to

$$\begin{aligned}&\mathrm{Ch}_\rho^{E_6/D_4}(\mathbf{s},\nu_\Psi,\nu_H,\tau)\\ &\quad= \mathrm{Tr}_{\rho,s_1}\Big((-1)^{s_2(F_\psi+F_H)}\,q^{L-\frac{1}{12}}\,\exp(2\pi i\nu_\Psi F_\Psi+2\pi i\nu_H F_H/\sqrt{3})\Big)_{H+\Psi}\,.\end{aligned} \tag{A.2}$$

Note the dual role played by the F_H operator here: $F_H/\sqrt{3}$ is the J-charge while F_H itself is the fermion number due to H-related degrees of freedom. Similarly, F_Ψ is both the fermion number for the two transverse fermions and also the helicity (or rather the Ψ-dependent part of the helicity).

The purpose of this Appendix is to use the Riemann identity (A.1) (or rather its complex conjugate) to establish the identity between the right hand sides of eqs. (2.13) and (2.17). Let us therefore start with eq. (2.13) and factorize the trace over the internal SCFT into a trace over the H-related part and a trace over the rest:

$$\begin{aligned}\mathcal{B}_a(\tau,\bar\tau) &= \sum_{\rho=0,\pm1}\mathrm{Tr}_\rho\Big(\Big(T_{(a)}^2-\tfrac{k_a}{8\pi\tau_2}\Big)\,q^{L-\frac{11}{12}}\,\bar q^{\bar L-\frac{1}{3}}\Big)_{(22,8)}\\ &\quad\times\frac{2}{|\eta(\tau)|^4}\sum_{\text{even s}}(-1)^{s_1+s_2}\,\frac{\partial Z_\Psi(\mathbf{s},\bar\tau)}{2\pi i\partial\bar\tau}\,\mathrm{Tr}_{\rho,s_1}\Big((-1)^{s_2F}\,\bar q^{\bar L-\frac{1}{24}}\Big)_H\,.\end{aligned} \tag{A.3}$$

Clearly, the expression on the second line here is universal for all the spacetime-

supersymmetric vacua; it is this expression that we are now going to rewrite in terms of the characters of the E_6/D_4 coset.

The E_6/D_4 coset is comprised of the H-boson and of an $SO(2)$ generated by the two transverse fermions; the Z_Ψ in eqs. (2.13) and (A.3) is precisely the partition function of those fermions. The derivative $\partial Z_\Psi/\partial\bar\tau$ can also be obtained from the $SO(2)$ characters $\mathrm{Ch}^{SO(2)}(\mathbf{s},\nu_\Psi,\bar\tau)$, which satisfy differential equations

$$\left(\frac{\partial}{\partial\bar\tau} - \frac{i}{4\pi}\frac{\partial^2}{\partial\nu_\Psi^2} + \frac{\partial\log\eta(\bar\tau)}{\partial\bar\tau}\right)\mathrm{Ch}^{SO(2)}(\mathbf{s},\nu_\Psi,\bar\tau) \;=\; 0. \tag{A.4}$$

Since for $\nu=0$ the characters are the same as the partition functions, we can express the $\partial Z_\Psi/\partial\bar\tau$ factors in eq. (A.3) in terms of the $SO(2)$ characters and their $\partial^2/\partial\nu_\Psi^2$ derivatives. Combining the result with the trace over the H-boson sector, we see that the expression on the second line of eq. (A.3) equals to

$$\frac{(1/4\pi^2)}{|\eta(\tau)|^4}\left(\frac{\partial^2}{\partial\nu_1^2} + 4\pi i\,\frac{\partial\log\eta(\bar\tau)}{\partial\bar\tau}\right)\sum_{\text{even s}}(-1)^{s_1+s_2}\;\mathrm{Ch}_\rho^{E_6/D_4}(\mathbf{s},\nu_\Psi,\nu_H,\bar\tau)\bigg|_{\nu_\Psi=\nu_H=0.} \tag{A.5}$$

The sum here is over the even sectors $\mathbf{s}$ only, but we can extend it to all the sectors since for $\nu_\Psi=\nu_H=0$, the Ramond-Ramond character vanishes together with its diagonal second derivatives. In this manner, we arrive at precisely the character sum on the left hand side of eq. (A.1). Now we can use the Riemann identity and relate everything to the Ramond-Ramond characters, but we still need to apply the differential operator in (A.5), which gives us

$$\frac{\sqrt3}{4\pi^2|\eta(\tau)|^4}\,\frac{\partial}{\partial\nu'_\Psi}\,\frac{\partial}{\partial\nu'_H}\,\mathrm{Ch}_\rho^{E_6/D_4}(RR,\nu'_\Psi,\nu'_H,\bar\tau)\bigg|_{\nu'_\Psi=\nu'_H=0}\,. \tag{A.6}$$

Here we have again used the vanishing of the Ramond-Ramond character and its diagonal second derivatives at $\nu'_\Psi=\nu'_H=0$.

To calculate the remaining derivatives in eq. (A.6), we factorize the E_6/D_4 character into the $SO(2)$ character times the character of the H-boson sector and apply the $\partial/\partial\nu'$ derivatives accordingly. For the $SO(2)$ character we have

$$\frac{\partial}{\partial\nu_\Psi}\,\mathrm{Ch}^{SO(2)}(RR,\nu_\Psi,\bar\tau)\bigg|_{\nu_\Psi=0} \;=\; 2\pi i\,\eta^2(\bar\tau). \tag{A.7}$$

Thus, the entire complicated expression on the second line of eq. (A.3) reduces to

$$\frac{i\sqrt3}{2\pi\eta^2(\tau)}\,\frac{\partial}{\partial\nu_H}\,\mathrm{Ch}_\rho^H(RR,\nu_H,\bar\tau)\bigg|_{\nu_H=0} \;=\; \frac{-1}{\eta^2(\tau)}\,\mathrm{Tr}_{\rho,R}\left(F\,(-1)^{F+\frac32}\,\bar q^{\bar L-\frac1{24}}\right)_H. \tag{A.8}$$

Finally, we substitute eq. (A.8) into eq. (A.3), combine the traces over the H-boson sector and over the model-dependent $c=(22,8)$ sector and identify the fermion number F of the H-boson sector with the $\bar F$ of the entire anti-holomorphic side of the internal SCFT. The result is

$$\mathcal{B}_a(\tau,\bar\tau) \;=\; \frac{-1}{\eta^2(\tau)}\,\mathrm{Tr}_{\bar R}\left((-)^{\overline{F}-\frac32}\,\overline{F}\;\left(T_{(a)}^2 - \tfrac{k_a}{8\pi\tau_2}\right)q^{L-\frac{11}{12}}\,\bar q^{\bar L-\frac38}\right)_{\text{int}} \tag{2.17}$$

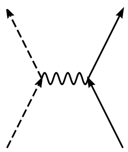
APPENDIX B
Moduli Dependence of Matter-Field Metrics in Orbifolds.

In this Appendix, we derive eqs. (3.7) and calculate the exponents q_I^i in terms of the orbifold parameters of the respective matter fields Q^I. Generally, in order to derive the parameters of the low-energy EQFT from the string theory, one calculates scattering amplitudes using either the string theory or the EQFT and demands that the two amplitudes for the same physical process agree with each other in the low-energy limit. The problem at hand involves the moduli dependence of the tree-level $Z_{\bar IJ}$ matrix for the matter fields, so we are going to calculate the tree-level four-particle amplitudes $\mathcal{A}^{(0)}(\overline{M}^\imath,\overline{Q}^{\bar I},Q^J,M^j)$ for the scattering of the moduli scalars M^j off the matter scalars Q^J.

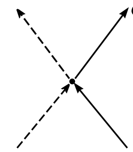
In EQFT, tree-level modulus-matter scattering is due to Einsteinian gravity and also due to sigma-model-like interactions arising from the moduli dependence of the $Z_{\bar{I}J}(M,\overline{M})$ matrix. Thus, following a similar calculation in ref. 34, we have

$$\mathcal{A}^{(0)}_{\mathrm{EQFT}}(\overline{M}^{i},\overline{Q}^{\bar{I}},Q^{J},M^{j}) \tag{B.1}$$

$=$



$+$



$+$

$$= \kappa^2 \frac{su}{t}\, Z_{\bar{I}J} G_{\bar{\imath}j} \quad + s\, \frac{\partial^2 Z_{\bar{I}J}}{\partial \overline{M}^{i}\, \partial M_{j}} \quad - s\, \frac{\partial Z}{\partial \overline{}}$$

where $s=-(k_1+k_2)^2$, $t=-(k_2+k_3)^2$ and $u=-(k_1+k_3)^2$ are Mandelstam's kinematic variables and $G_{\bar{\imath}j}=\kappa^{-2}\partial_{\bar{\imath}}\partial_j\hat{K}$ is the metric of the moduli space.

Later in this Appendix, we will show that for the factorizable orbifolds, the tree-level string-theoretical amplitudes for scattering of the untwisted moduli off the matter scalars look like

$$\mathcal{A}^{(0)}_{\mathrm{string}}(\overline{M}^{i},\overline{Q}^{\bar{I}},Q^{J},M^{j}) \;=\; \kappa^2\delta_{\bar{\imath}j}\delta_{\bar{I}J}\left(\frac{su}{t} \;+\; sq^{j}_{J} \;+\; O(\alpha' k^4)\right). \tag{B.2}$$

At this point, however, we would like to derive the eq. (3.7) from eqs. (B.1) and (B.2) before we proceed to derive the eq. (B.2) itself.

As written, the amplitudes (B.1) and (B.2) assume different normalization conventions for the external particles: The string-theoretical amplitude (B.2) assumes them to be canonically normalized while the field-theoretical amplitude (B.1) assumes that the particles are normalized exactly as the fields $\overline{M}^{i}$, $\overline{Q}^{\bar{I}}$, Q^J and M^i. Translating the string amplitude into the field-theoretical normalization conventions gives

$$\kappa^2 G_{\bar{\imath}j} Z_{\bar{I}J}\left(\frac{su}{t} \;+\; sq^{j}_{J} \;+\; O(\alpha' k^4)\right), \tag{B.3}$$

and it this formula that should agree in the low-energy limit with the field-

theoretical amplitude (B.1). By inspection of the two amplitudes (B.1) and (B.3), their low-energy limits are similar and the only not-trivial requirement of complete agreement has to do with the terms proportional to the Mandelstam's s. Comparing their coefficients, we arrive at

$$\begin{aligned}\partial_{\bar{\imath}}\left(Z^{-1}\partial_j Z\right)^{I}_{\ J} \;&\equiv\; \left(Z^{-1}\partial_{\bar{\imath}}\partial_j Z - Z^{-1}\partial_{\bar{\imath}}Z\, Z^{-1}\partial_j Z\right)^{I}_{\ J}\\ &=\; \kappa^2 G_{\bar{\imath}j}\delta^{I}_{J}\, q^{j}_{J} = \frac{\delta_{\bar{\imath}j}\delta^{I}_{J}q^{j}_{J}}{(M^j+\overline{M}^j)^2}\,;\end{aligned} \tag{B.4}$$

the last equality here follows from eq. (3.5).

It remains but to solve the differential equations (B.4) for the moduli dependence of the $Z_{\bar{I}J}$ matrix. It is easy to see that a generic solution of these equations looks like

$$Z_{\bar{I}J}(M,\overline{M}) \;=\sum_{L} F^{*}_{\bar{I}L}(\overline{M})F_{JL}(M)\,\prod_{i}\left(M^i+\overline{M}^i\right)^{-q^{i}_{L}} \tag{B.5}$$

where $F_{JL}(M)$ is an arbitrary non-degenerate matrix of holomorphic functions of the moduli. This arbitrariness reflect our freedom to use arbitrary (but holomorphic) moduli-dependent coordinates for the space of the matter fields Q^I. Without loss of generality, we may choose those coordinates such that $F_{JL}(M)\equiv\delta_{JL}$; with this choice, eq. (B.5) reduces to the eq. (3.7).

Let us now turn to the derivation of the string-theoretical tree-level scattering amplitude (B.2); we use the formalism and many explicit results of refs. 66. In the Hamiltonian formalism for the world-sheet quantities, we have

$$\mathcal{A}^{(0)}_{\mathrm{string}}(\overline{M}^{i},\overline{Q}^{I},Q^{J},M^{j}) \;=\; g^2_{\mathrm{string}}\int d^2z\,\left\langle Q^I\middle|\,T\left(V_{\overline{M}^i}(w),V_{M^i}(z)\right)\middle|Q^J\right\rangle, \tag{B.6}$$

where $\left|Q^J\right\rangle$ and $\left\langle Q^I\right|$ are the asymptotic initial and final states of the matter scalar and T is the "time"-ordered product of the vertex operators for the moduli.

For the untwisted moduli of a factorizable orbifold, these operators are

$$\begin{aligned}
V_{T^j} &= \exp(ik_\mu X^\mu)\,\partial\overline{X}^j\,\left(\bar\partial X^j + \tfrac{1}{2}k_\mu\psi^\mu\psi^j\right),\\
V_{\overline{T}^i} &= \exp(ik_\mu X^\mu)\,\partial X^i\,\left(\bar\partial\overline{X}^i + \tfrac{1}{2}k_\mu\psi^\mu\overline{\psi}^i\right),\\
V_{U^j} &= \exp(ik_\mu X^\mu)\,\partial X^j\,\left(\bar\partial X^j + \tfrac{1}{2}k_\mu\psi^\mu\psi^j\right),\\
V_{\overline{U}^i} &= \exp(ik_\mu X^\mu)\,\partial\overline{X}^i\,\left(\bar\partial\overline{X}^i + \tfrac{1}{2}k_\mu\psi^\mu\overline{\psi}^i\right)
\end{aligned} \tag{B.7}$$

(in string units $\alpha' = \frac{1}{2}$). For the sake of notational simplicity, we are going to consider the $(1,1)$ moduli T^i first and only then address the $(1,2)$ moduli U^i.

We begin by explaining the $\delta_{\bar\imath j}\delta_{\bar I J}$ factor in eq. (B.2) (for the $(1,1)$ moduli), which means that the scattering process $Q^J + T^j \to Q^I + T^i$ is possible only if $i = j$ and $I = J$. Clearly, Q^J and Q^I must belong to the same twist sector of the orbifold, and if that sector is twisted, they must originate in the same fixed point (or fixed sub-torus) of the internal six-torus. For any given twisted sector of any supersymmetric orbifold, all the matter fields arising from that sector have identical structures as far as the right-moving world-sheet degrees of freedom are concerned. Similarly, all the matter fields arising from the completely untwisted sector have the same trivial structure with regard to the left-moving ∂X^i and $\partial\overline{X}^i$. Hence, either the right-moving or the left-moving creation/annihilation operators contained in the moduli vertex V_{T^j} have to cancel against those in the other vertex $V_{\overline{T}^i}$, and this is possible only if $i = j$. On the other hand, if $i = j$, then the operator product $T\left(V_{\overline{T}^i}(w), V_{T^j}(z)\right)$ is completely diagonal with respect to all the massless matter scalars, so we must have $I = J$ as well.

Next, consider the diagonal matrix elements $\left\langle Q^I\middle| T\left(V_{\overline{T}^i}, V_{T^i}\right)\middle|Q^I\right\rangle$. Like all vertex correlators on the spherical world sheet, these matrix elements are products of the holomorphic and the antiholomorphic factors corresponding to the two world-sheet chiralities. On the left-moving, bosonic side of the heterotic string, we have

$$\begin{aligned}
&\left\langle Q^I\middle| T\left(V_{\overline{T}^i}(w,\bar w), V_{T^i}(z,\bar z)\right)\middle|Q^I\right\rangle_{\text{left}} \\
&\qquad = w^{-s/8}z^{-u/8}(z-w)^{-t/8}\left\langle Q^I\middle| T\left(\partial X^i(w), \partial\overline{X}^i(z)\right)\middle|Q^I\right\rangle.
\end{aligned} \tag{B.8}$$

Let $2\pi\eta^i_I$ $(0 \le \eta^i_I < 1)$ be the angle by which the ith internal plane is rotated in the twist sector giving rise to the matter particle Q^I. Then in that sector, the operators ∂X^i and $\partial\overline{X}^i$ decompose according to

$$\begin{aligned}
\partial\overline{X}^i(z) &= \sum_{n=-\infty}^{+\infty}\bar\alpha^i_{n+\eta}z^{-1-n-\eta},\\
\partial X^i(w) &= \sum_{m=-\infty}^{+\infty}\alpha^i_{-m-\eta}w^{-1+m+\eta},
\end{aligned} \tag{B.9}$$

where $[\bar\alpha^i_{n+\eta}, \alpha^i_{-m-\eta}] = (n+\eta)\delta_{nm}$ and $\eta \equiv \eta^i_I$. Generally, the state $\left|Q^I\right\rangle$ may have $N^i_I \ge 0$ quanta of the $(\bar\alpha_\eta, \alpha_{-\eta})$ oscillator or $\overline{N}^i_I \ge 0$ quanta of the $(\alpha_{1-\eta}, \bar\alpha_{\eta-1})$ oscillator,* but it must be the ground state of all the other oscillators of the ith plane. Therefore,

$$\left\langle Q^I\middle| T\left(\partial X^i(w), \partial\overline{X}^i(z)\right)\middle|Q^I\right\rangle = w^{\eta-1}z^{-\eta}\left[\frac{(1-\eta)w + \eta z}{(z-w)^2} + \frac{\eta}{z}N^i_I + \frac{(1-\eta)}{w}\overline{N}^i_I\right] \tag{B.10}$$

(again, $\eta \equiv \eta^i_I$); substituting this matrix element into eq. (B.8) and integrating by parts, we arrive at

$$\begin{aligned}
\left\langle Q^I\middle| T\left(V_{\overline{T}^i}(w,\bar w), V_{T^i}(z,\bar z)\right)\middle|Q^I\right\rangle_{\text{left}} &= -w^{\eta-1-s/8}z^{-\eta-u/8}(z-w)^{-1-t/8}\\
&\times\left[\frac{u+(1-\eta)t}{8+t} + \frac{\eta t}{8\eta+u}N^i_I - \frac{(1-\eta)t}{8(1-\eta)+s}\overline{N}^i_I\right]
\end{aligned} \tag{B.11}$$

plus a total-holomorphic-derivative term that would not contribute to the integral (B.6).

$\star$ In the untwisted sector and in the twisted sectors with $\eta^i = 0$, all matter particles have $N^i_I = \overline{N}^i_I = 0$. In other twisted sectors, all particles have either $\overline{N}^i_I = 0$ and $\eta^i_I N^i_I < 1$ or $N^i_I = 0$ and $(1-\eta^i_I)\overline{N}^i_I < 1$.

Now consider the right-moving, supersymmetric side of the heterotic string, where we have

$$\begin{aligned}\left\langle Q^I\right| T\big(V_{\overline{T}^i}(w,\bar w), V_{T^i}(z,\bar z)\big)\left|Q^I\right\rangle_{\rm right} \;&=\; \bar w^{-s/8}\bar z^{-u/8}(\bar z-\bar w)^{-t/8}\\ &\times\Big[\left\langle Q^I\right| T\left(\bar\partial\overline{X}^i(\bar w),\bar\partial X^i(\bar z)\right)\left|Q^I\right\rangle\\ &\quad+\;\frac{t}{8}\left\langle Q^I\right| T\left(\psi^\mu(\bar w),\psi^\mu(\bar z)\right)\left|Q^I\right\rangle\left\langle Q^I\right| T\left(\overline{\psi}^i(\bar w),\psi^i(\bar z)\right)\left|Q^I\right\rangle\Big].\end{aligned}\tag{B.12}$$

Clearly, the bosonic matrix elements on the right hand side here is simply the complex conjugate of eq. (B.10) for the special case $N_I^i=\overline{N}_I^i=0$,

$$\left\langle Q^I\right| T\left(\bar\partial\overline{X}^i(w),\bar\partial X^i(z)\right)\left|Q^I\right\rangle \;=\; \bar w^{\eta-1}\bar z^{-\eta}\,\frac{(1-\eta)\bar w+\eta\bar z}{(\bar z-\bar w)^2}\,.\tag{B.13}$$

The fermionic matrix elements in eq. (B.12) follow from the decomposition

$$\begin{aligned}\psi^i(z)\;&=\;\sum_{r=-\infty}^{+\infty}\psi^i_{r+\eta}z^{-\frac12-r-\eta},\\ \overline{\psi}^i(w)\;&=\;\sum_{p=-\infty}^{+\infty}\overline{\psi}^i_{-p-\eta}w^{-\frac12+p+\eta},\end{aligned}\tag{B.14}$$

where $\{\psi^i_{r+\eta},\overline{\psi}^i_{-p-\eta}\}=\delta_{rp}$ and — in the sector containing the Q^I — r and p are half-integers and $\eta\equiv\eta_I^i$. For any of the twisted sectors of the orbifold, all matter states $\left|Q^I\right\rangle$ are annihilated by all the $\psi^i_{r+\eta}$ with $r\geq\frac12$ and all the $\overline{\psi}^i_{-p-\eta}$ with $-p\geq\frac12$ (because of the GSO projection, this is true even for the $\overline{\psi}^i_{\frac12-\eta}$ when $\frac12-\eta_I^i<0$). In the completely untwisted sector, however, the right-moving sides of the matter states are formed according to $\psi^\ell_{-\frac12}\left|0\right\rangle$, so one should distinguish between the cases $\ell_I=i$ and $\ell_I\neq i$. Therefore,

$$\left\langle Q^I\right| T\left(\overline{\psi}^i(\bar w),\psi^i(\bar z)\right)\left|Q^I\right\rangle=\frac{-1}{\bar z-\bar w}\times\begin{cases}(\bar w/\bar z)^{\eta_I^i} & \text{for all twisted states,}\\ 1 & \text{for untwisted states with } \ell_I\neq i,\\ (\bar w/\bar z) & \text{for untwisted states with } \ell_I=i;\end{cases}\tag{B.15}$$

similarly,

$$\left\langle Q^I\right| T\left(\psi^\mu(\bar w),\psi^\mu(\bar z)\right)\left|Q^I\right\rangle \;=\; \frac{-1}{\bar z-\bar w}\,.\tag{B.16}$$

Substituting the matrix elements (B.13), (B.15) and (B.16) into eq. (B.12) and integrating by parts, we arrive at

$$\begin{aligned}&\left\langle Q^I\right| T\big(V_{\overline{T}^i}(w,\bar w), V_{T^i}(z,\bar z)\big)\left|Q^I\right\rangle_{\rm right}\\ &\quad=\;\bar w^{\eta-1-s/8}\bar z^{-\eta-u/8}(\bar z-\bar w)^{-1-t/8}\\ &\quad\quad\times\begin{cases}-s^2/8u & \text{for untwisted states with } \ell_I=i,\\ +s/8 & \text{for all other states,}\end{cases}\end{aligned}\tag{B.17}$$

plus a total-antiholomorphic-derivative term.

At this point, all we need to do is to substitute eqs. (B.11) and (B.17) into eq. (B.6) and perform the d^2z integral. Taking the low-energy limit $|s|,|t|,|u|\ll1$ of the resulting expression, we finally arrive at

$$\mathcal{A}^{(0)}_{\rm string}(\overline{T}^i,\overline{Q}^I,Q^J,T^j)\;=\;\tfrac18 g^2_{\rm string}\,\delta_{ij}\delta_{IJ}\left(\frac{su}{t}\;+\;s\,q_I^i\;+\;s\,O(s,t,u)\right).\tag{B.18}$$

where

$$q_I^i\;=\;\begin{cases}(1-\eta_I^i)+N_I^i-\overline{N}_I^i & \text{for all twisted states,}\\ 0 & \text{for untwisted states with } \ell_I\neq i,\\ 1 & \text{for untwisted states with } \ell_I=i.\end{cases}\tag{B.19}$$

Note that eq. (B.18) is written in string units $\alpha'=\frac12$; translating it into the conventional units gives us eq. (B.2) for the $(1,1)$ moduli.

The above arguments leading to eq. (B.18) can be applied almost verbatim to the scattering amplitudes involving the $(1,2)$ moduli U^i instead of the $(1,1)$ moduli T^i. In light of the vertex operators (B.7), the right-moving degrees of freedom do not distinguish between the two kinds of the untwisted moduli, so the

right-moving matrix element (B.17) is exactly the same in both cases. However, on the left-moving (bosonic) side of the heterotic string, replacing T^i with U^i and $\overline{T}^i$ with $\overline{U}^i$ requires an interchange between the ∂X^i and the $\partial \overline{X}^i$. Hence, in the left-moving matrix element (B.11), one should interchange N_I^i with $\overline{N}_I^i$ and η_I^i with $(1-\eta_I^i)$, unless $\eta_I^i = 0$, in which case there are no modifications at all. Actually, if a factorizable orbifold does have an unfrozen $(1,2)$ modulus U^i, the only allowed values of η_I^i are 0 and $\frac{1}{2}$[⋆] and so the interchange $\eta_I^i \leftrightarrow (1-\eta_I^i)$ is never needed. Therefore,

$$\mathcal{A}^{(0)}_{\text{string}}(\overline{U}^i, \overline{Q}^I, Q^J, U^j) \;=\; \tfrac{1}{8} g^2_{\text{string}}\, \delta_{ij}\delta_{IJ} \left(\frac{su}{t} \;+\; s\,\tilde{q}_I^i \;+\; s\,O(s,t,u) \right). \quad \text{(B.20)}$$

where $\tilde{q}_I^i$ is exactly as in eq. (B.19), except for the interchange $N_I^i \leftrightarrow \overline{N}_I^i$.

Most matter fields Q^I have $\tilde{q}_I^i = q_I^i$ and thus contribute equally to the "modular anomaly" coefficients α_a^i (*cf.* eq. (3.10)) for the $(1,2)$ modulus U^i and the $(1,1)$ modulus T^i of the same internal plane. The only exception to this rule are the Q^I for which $\eta_I^i = \frac{1}{2}$ and either $N_I^i = 1$ and $\overline{N}_I^i = 0$ or else $N_I^i = 0$ and $\overline{N}_I^i = 1$. When the orbifold group never twists the ith plane by any angle other than zero or π, such states always come in pairs: They have identical gauge and other quantum numbers and the only difference between the two members of a pair is $N_I^i \leftrightarrow \overline{N}_I^i$. Hence, one member of a pair has $q_I^i = \frac{3}{2}$ and $\tilde{q}_I^i = -\frac{1}{2}$ while the other has $q_I^i = -\frac{1}{2}$ and $\tilde{q}_I^i = \frac{3}{2}$ and the net contribution of such a pair to the coefficients α_a^i is zero — again for either of the two moduli of the ith plane.

Actually, the full story of such pairs of matter field is more complicated since they allow for non-diagonal scattering amplitudes in which one member of a pair turns into the other member while the U^i modulus turns into the T^i or vice verse. Therefore, we should add some non-diagonal terms to the right-hand

⋆ If any sector of the orbifold were to rotate the ith internal plane by any angle other than zero or π, the value of the U^i modulus of the internal torus would be completely frozen by the orbifolding procedure.

side of eq. (B.4), and the resulting moduli-dependent matrix $Z_{\bar{I}J}$ is not quite as diagonal as in eq. (3.7). Instead, for the $N_I^i \leftrightarrow \overline{N}_I^i$ pairs of matter fields — and only for such pairs, the rest of the matrix $Z_{\bar{I}J}(M,\overline{M})$ is exactly as in eq. (3.7), — we have inextricably entangled 2×2 blocks of rather complicated moduli dependence. However, the determinants of such 2×2 blocks satisfy exactly the same differential equations as if the non-diagonal scattering amplitudes did not exist. Consequently, while the moduli transformation rules for the $N_I^i \leftrightarrow \overline{N}_I^i$ pairs are more complicated than eq. (3.8), their effect upon the moduli anomalies of the Wilsonian gauge couplings is exactly as in eq. (3.9), with the net contribution of each pair to the α_a^i being exactly as if the pair had $q_I^i = (\frac{3}{2}, -\frac{1}{2})$. Specifically, the net contribution of a $N_I^i \leftrightarrow \overline{N}_I^i$ pair of matter fields to the α_a^i is exactly zero. This completes the proof that *whenever a factorizable orbifold has an unfrozen* $(1,2)$ *modulus* U^i, *that modulus has exactly the same modular anomaly coefficients* α_a^i *as the* $(1,1)$ *modulus* T^i *of the same internal plane.*

APPENDIX C
Examples of Factorizable Orbifolds

In this Appendix, we verify eqs. (3.22) for all factorizable $N=2$ orbifolds and for several examples of factorizable $N=1$ orbifolds. To save space, we present only a dozen of the $N=1$ orbifolds here, but we have actually investigated many more, and for all those orbifolds we found eqs. (3.22) holding true for all three internal planes and all the gauge couplings of the orbifold.

C.1. Factorizable $N=2$ Orbifolds.

A supersymmetric orbifold has unbroken $N=2$ spacetime supersymmetry when the orbifold group D never rotates one of the internal planes. In the notations of eq. (3.19), for a factorizable orbifold of this kind, one has $D_1 = D$ and $\mathbf{b}_a^{N=2}(1) = \mathbf{b}_a$ for the untwisted first plane while for the other two planes,

$D_2 = D_3 = 1$ and $\mathbf{b}_a^{N=2}(2) = \mathbf{b}_a^{N=2}(3) = 0$. Let us now demonstrate that for all such orbifolds, one also has

$$\alpha_a^1 \ = \ \mathbf{b}_a \qquad \text{and} \quad \alpha_a^2 \ = \ \alpha_a^3 \ = \ 0. \tag{C.1}$$

This would not only confirm eqs. (3.22) for all three planes and all the gauge couplings but also show that $\delta_{\rm GS}^1 = \delta_{\rm GS}^2 = \delta_{\rm GS}^3 = 0$.

The $N = 2$ spacetime supersymmetry has two kinds of massless supermultiplets containing scalar particles, namely the vector multiplets and the hypermultiplets. In $N = 2$ orbifolds of the heterotic string, scalars belonging to vector supermultiplets include the dilaton S, the two moduli T^1 and U^1 of the untwisted plane, and the matter fields Q^I originating in completely untwisted string states with $\ell_I = 1$; all other moduli and matter scalars belong to hypermultiplets. For the matter scalars, this distinction parallels eq. (B.19) for $i = 1$: All matter scalars arising from twisted states have $\eta_I^1 = 0$ and hence $q_I^1 = 0$, the untwisted states with $\ell_i \neq 1$ also have $q_I^1 = 0$, but the untwisted states with $\ell_I = 1$ have $q_I^1 = 1$. Therefore, eqs. (3.10) for $i = 1$ reduce to

$$\begin{aligned} \alpha_a^1 \ &= \sum_I^{\rm hyper} T_a(Q^I) \ - \sum_I^{\rm vector} T_a(Q^I) \ - \ T(G_a) \\ &= \sum_I^{\rm hyper} T_a(Q^I) \ - \ 2T(G_a) \ \equiv \ \mathbf{b}_a \quad (\text{for } N=2). \end{aligned} \tag{C.2}$$

The fact that all the Q^I belonging to hypermultiplets have $q_i^1 = 0$ and thus have metrics $Z_{\bar{I}J}$ that do not depend on T^1 and U^1 reflects a universal property of $N = 2$ EQFTs: The metric for the hypermultiplets does not depend on the vector superfields and vice verse. Thus, since the moduli $T^{2,3}$ and $U^{2,3}$ belong to hypermultiplets, we should also have $q_I^2 = q_I^3 = 0$ for Q^I in vector supermultiplets, which is indeed the case according to eq. (B.19). On the other hand, the

metric matrix for the hypermultiplets may depend on the $T^{2,3}$ and $U^{2,3}$, but the resulting field space should have a quaternionic geometry. Consequently, for any two matter scalars Q^1 and $\overline{Q}^2$ belonging to the same hypermultiplet, $q_1^i + q_2^i = 1$ for $i = 2, 3$; because of our present focus on the orbifolds, we prefer to derive this result from eqs. (B.19) instead of going through a more general $N = 2$ analysis.

Consider a matter hypermultiplet $(Q^1, \overline{Q}^2)$ arising from a twisted sector. In $N = 1$ terms, the two chiral supermultiplets Q^1 and Q^2 arise from oppositely twisted sectors; hence, in eqs. (B.19), one should use $\eta_2^i = 1 - \eta_1^i$ for $i = 2, 3$. Similarly, $N_2^i = \overline{N}_1^i$ and $\overline{N}_2^i = N_1^i$; thus, for the twisted hypermultiplets, eqs. (B.19) give $q_1^i + q_2^i = 1$ for $i = 2, 3$. On the other hand, for the untwisted hypermultiplets, both Q^1 and Q^2 arise from the same untwisted sector. However, if Q^1 has $\ell_1 = 2$ then Q^2 has $\ell_2 = 3$ and vice verse, if $\ell_1 = 3$ then $\ell_2 = 2$. Thus, according to eqs. (B.19), we again have $q_1^i + q_2^i = 1$ for $i = 2, 3$. Since Q^1 and Q^2 always have exactly opposite gauge quantum numbers, $T_a(Q^1) = T_a(Q^2)$ for all a. Therefore, according to eqs. (3.10), the net contribution of any matter hypermultiplet to the modular anomalies $\alpha_a^{2,3}$ is precisely zero. Consequently,

$$\alpha_a^2 \ = \ \alpha_a^3 \ = \sum_I^{\rm vector} T_a(Q^I) \ - \ T(G_a) \ = \ 0. \tag{C.3}$$

Eqs. (C.3) do not merely confirm eqs. (3.22) for the moduli of the twisted planes, they also verify a stronger string-EQFT consistency condition required by the unbroken $N = 2$ spacetime supersymmetry. Specifically, in a locally $N = 2$ supersymmetric EQFT, the gauge couplings are not allowed to depend on any hypermultiplets, and any mixing between the hypermultiplets and the dilaton S (which belongs to a vector supermultiplet) is also forbidden. Thus, at the one-loop level, we must have $\alpha_a^2 = \alpha_a^3 = 0$ and also $\delta_{\rm GS}^2 = \delta_{\rm GS}^3 = 0$, and both requirements are indeed upheld by eq. (C.3). ($\delta_{\rm GS}^2 = \delta_{\rm GS}^3 = 0$ follows from eq. (3.22) and the fact that $\mathbf{b}_a(i = 2, 3) \equiv 0$ for an $N = 2$ orbifold.)

On the other hand, the fact that eq. (3.22) for the untwisted first plane holds for $\delta^1_{\rm GS}=0$ does not have any profound significance from the $N=2$ point of view. Both the dilaton S and the moduli T^1 and U^1 of the untwisted plane belong to vector supermultiplets of the $N=2$ supersymmetry, so there is no reason why they should not mix with each other in the perturbative string theory. In fact, they do mix with each other: According to eq. (3.20),

$$V^{(1)}_{N=2}(M,\overline{M})\ =\ \Omega(T^1,U^1,\overline{T}^1,\overline{U}^1), \tag{C.4}$$

where Ω obtains from an explicit calculation of the entire string-theoretical threshold corrections $\Delta_a(M,\overline{M})$ rather than the differences $\Delta_a-\Delta_{a'}$. The explicit form of this mixing is given by eq. (3.21); the derivation of this formula and its physical implications will be presented in a forthcoming article.

C.2. Z_3 Orbifolds.

The orbifold group of the $N=1$ supersymmetric Z_3 orbifolds[43] is generated by the rotation $\Theta=(e^{2\pi i/3},e^{2\pi i/3},e^{2\pi i/3})$. There are five inequivalent modular-invariant ways this group may act on the $E_8\times E_8$ degrees of freedom; hence, there are five distinct Z_3 orbifolds, with unbroken gauge symmetries being respectively $E_8\times E_8$, $(E_6\times SU(3))\times E_8$, $(E_6\times SU(3))\times(E_6\times SU(3))$, $(SO(14)\times U(1))\times SU(9)$ and $(SO(14)\times U(1))\times(E_7\times U(1))$.

A Z_3 orbifold has no $(1,2)$ moduli but nine $(1,1)$ moduli, and for generic values of these moduli, the orbifold is not quite factorizable — the three internal planes are not mutually orthogonal but mix with each other. However, since the purpose of this Appendix is to present examples of factorizable orbifolds, we impose factorizability by fiat, *i.e.*, we assume that all six of the off-diagonal $(1,1)$ moduli have zero values and concentrate on the way the gauge couplings depends on the three diagonal moduli $T^{1,2,3}$. Specifically, we are going to calculate the

modular anomaly coefficients (3.10) and show that

$$\alpha^i_a\ =\ k_a\delta_{\rm GS}, \tag{C.5}$$

in full agreement with eq. (3.22) for orbifolds without $N=2$ supersymmetric sectors. Because of the obvious symmetry, the coefficient $\delta_{\rm GS}$ in eq. (C.5) is always the same for all three internal planes of a Z_3 orbifold; however, the Z_3 orbifolds with different gauge groups generally have different values of $\delta_{\rm GS}$.

Let us start with the left-right symmetric Z_3 orbifold whose gauge group is $G=E_6\times SU(3)\times E_8$. The matter states for this orbifold are summarized in the following table

sector	$E_6\times SU(3)\times E_8$	#	ℓ_I or η^i_I	oscillators	average q^i_I
untwisted	$(\mathbf{27},\mathbf{3},\mathbf{1})$	3	$\ell=1,2,3$	none	$(\frac13,\frac13,\frac13)$
Θ	$(\mathbf{27},\mathbf{1},\mathbf{1})$	27	$\eta=(\frac13,\frac13,\frac13)$	none	$(\frac23,\frac23,\frac23)$
Θ	$(\mathbf{1},\bar{\mathbf{3}},\mathbf{1})$	81	$\eta=(\frac13,\frac13,\frac13)$	$\sum_i N^i=1,\ \overline{N}^i=0$	$(1,1,1)$

where the last column gives the average value of the (q^1,q^2,q^3) (calculated according to eq. (B.19)) for all the states in any given raw. Substituting this spectrum — and the values of the q^i_I — into eqs. (3.10) and totalling the sums gives us

$$\alpha^i_{E(6)}\ =\ \alpha^i_{SU(3)}\ =\ \alpha^i_{E(8)}\ =\ -30. \tag{C.6}$$

Next consider the Z_3 orbifold with the completely unbroken $E_8\times E_8$ gauge group. This orbifold is somewhat peculiar since it has no untwisted matter fields at all. It does have 243 twisted matter fields, but all of them are singlets under the gauge group and thus do not contribute the modular anomalies of the gauge couplings. Therefore,

$$\alpha^i_{E(8)}\ =\ \alpha^i_{E(8)'}\ =\ -30. \tag{C.7}$$

The next Z_3 orbifold has both of the E_8 groups twisted in the same manner as the twisted E_8 of the left-right symmetric orbifold; its unbroken gauge group is

$G = E_6 \times SU(3) \times E_6 \times SU(3)$. The matter states of this orbifold are as follows:

sector	$E_6 \times SU(3) \times E_6 \times SU(3)$	#	ℓ_I or η_I^i	oscillators	average q_I^i
untwisted	$(\mathbf{27}, \mathbf{3}, \mathbf{1}, \mathbf{1})$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
untwisted	$(\mathbf{1}, \mathbf{1}, \mathbf{27}, \mathbf{3})$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
Θ	$(\mathbf{1}, \bar{\mathbf{3}}, \mathbf{1}, \bar{\mathbf{3}})$	27	$\eta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	none	$(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$

Therefore,

$$\alpha^i_{E(6)} = \alpha^i_{E(6)'} = \alpha^i_{SU(3)} = \alpha^i_{SU(3)'} = -3. \tag{C.8}$$

Note that for this orbifold eq. (C.5) holds true, but $\delta_{\rm GS} = -3$ rather than -30. We shall see momentarily that the other two Z_3 orbifolds in which both E_8 groups are broken also have $\delta_{\rm GS} \neq -30$.

Indeed, the Z_3 orbifold with the $G = SO(14) \times U(1) \times SU(9)$ gauge group has the following matter fields:

sector	$SO(14) \times U(1) \times SU(9)$	#	ℓ_I or η_I^i	oscillators	average q_I^i
untwisted	$(\mathbf{14}, -1, \mathbf{1})$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
untwisted	$(\mathbf{64}, +\frac{1}{2}, \mathbf{1})$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
untwisted	$(\mathbf{1}, 0, \mathbf{84})$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
Θ	$(\mathbf{1}, +\frac{2}{3}, \bar{\mathbf{9}})$	27	$\eta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	none	$(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$

where the $U(1)$ charges are normalized according to $k_{U(1)} = 2$. Substituting this table of matter fields into eq. (3.10), we obtain

$$\alpha^i_{SO(14)} = \alpha^i_{SU(9)} = \tfrac{1}{2}\alpha^i_{U(1)} = -3. \tag{C.9}$$

Similarly, for the remaining Z_3 orbifold with the $G = SO(14) \times U(1) \times E_7 \times U(1)$ gauge group, the matter fields are

sector	$SO(14) \times U(1) \times E_7 \times U(1)$	#	ℓ_I or η_I^i	oscillators	average q_I^i
untwisted	$(\mathbf{14}, +1, \mathbf{1}, 0)$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
untwisted	$(\mathbf{64}, +\frac{1}{2}, \mathbf{1}, 0)$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
untwisted	$(\mathbf{1}, 0, \mathbf{56}, +\frac{1}{2})$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
untwisted	$(\mathbf{1}, 0, \mathbf{1}, -1)$	3	$\ell = 1, 2, 3$	none	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
Θ	$(\mathbf{14}, -\frac{1}{3}, \mathbf{1}, +\frac{1}{3})$	27	$\eta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	none	$(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$
Θ	$(\mathbf{1}, +\frac{2}{3}, \mathbf{1}, -\frac{2}{3})$	27	$\eta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	none	$(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$
Θ	$(\mathbf{1}, +\frac{2}{3}, \mathbf{1}, +\frac{1}{3})$	81	$\eta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\sum_i N^i = 1,\ \overline{N}^i = 0$	$(1, 1, 1)$

and hence

$$\alpha^i_{SO(14)} = \alpha^i_{E(7)} = -12, \qquad \alpha^i_{U(1)} = -12 k_{U(1)} \tag{C.10}$$

where $k_{U(1)} = \binom{2\,0}{0\,1}$ is the normalization matrix for the two abelian gauge charges of the model. Again, eqs. (C.5) are satisfied but for $\delta_{\rm GS} = -12$ rather than -30.

C.3. $Z_2 \times Z_2$ Orbifolds.

The orbifold group of the $Z_2 \times Z_2$ orbifolds [43,67] is generated by two rotations, $\Theta_1 = (-1, -1, +1)$ and $\Theta_2 = (-1, +1, -1)$; consequently, there are six untwisted moduli, $T^{1,2,3}$ and $U^{1,2,3}$. Again, there are five inequivalent modular-invariant embeddings of the orbifold group into the $E_8 \times E_8$ Kac-Moody algebra and hence five distinct $Z_2 \times Z_2$ orbifolds, whose unbroken gauge symmetries are respectively $\big(E_6 \times U(1)^2\big) \times E_8$, $\big(E_6 \times U(1)^2\big) \times SO(16)$, $\big(E_6 \times U(1)^2\big) \times (SO(8) \times SO(8))$, $(SU(8) \times U(1)) \times (E_7 \times SU(2))$ and $(SU(8) \times U(1)) \times (SO(12) \times SU(2) \times SU(2))$.

All three internal planes of a $Z_2 \times Z_2$ orbifolds have non-trivial little groups $D_i = Z_2$ making non-trivial $N = 2$ orbifolds. Depending on a particular $Z_2 \times Z_2$ orbifold and on a particular plane, one may get either of the two Z_2 orbifolds:

The first has $G = E_7 \times SU(2) \times E_8$ and the hypermultiplet spectrum consisting of two copies of $(\mathbf{56},\mathbf{2},\mathbf{1})$, sixteen copies of $(\mathbf{56},\mathbf{1},\mathbf{1})$ and sixty four copies of $(\mathbf{1},\mathbf{2},\mathbf{1})$; consequently,

$$\mathbf{b}^{N=2}_{E(7)} \;=\; \mathbf{b}^{N=2}_{SU(2)} \;=\; +84, \qquad \mathbf{b}^{N=2}_{E(8)} \;=\; -60. \tag{C.11}$$

The second Z_2 orbifold has $G = E_7 \times SU(2) \times SO(16)$ and the hypermultiplet spectrum consisting of two copies of $(\mathbf{56},\mathbf{2},\mathbf{1})$, two copies of $(\mathbf{1},\mathbf{1},\mathbf{128})$ and sixteen copies of $(\mathbf{1},\mathbf{2},\mathbf{16})$; consequently

$$\mathbf{b}^{N=2}_{E(7)} \;=\; -12, \qquad \mathbf{b}^{N=2}_{SU(2)} \;=\; +180, \qquad \mathbf{b}^{N=2}_{SO(16)} \;=\; +36. \tag{C.12}$$

For the left-right symmetric $Z_2 \times Z_2$ orbifold, each of the three little groups D_i produces the $N = 2$ orbifold with $G^{N=2} = E_7 \times SU(2) \times E_8$ thus giving us eqs. (C.11) for the $\mathbf{b}^{N=2}_a(i)$. The $N = 1$ orbifold itself has $G = E_6 \times U(1)^2 \times E_8$ and the following massless matter fields Q^I:

sector	$E_6 \times U(1)^2 \times E_8$	#	ℓ_I or η^i_I	osc.	average q^i_I
untwisted	$(\mathbf{27},+\frac12,+\frac12,\mathbf{1})+(\mathbf{1},-\frac12,+\frac32,\mathbf{1})+\text{c.c}$	1	$\ell=1$	0	$(1,0,0)$
untwisted	$(\mathbf{27},-\frac12,+\frac12,\mathbf{1})+(\mathbf{1},+\frac12,+\frac32,\mathbf{1})+\text{c.c}$	1	$\ell=2$	0	$(0,1,0)$
untwisted	$(\mathbf{27},0,-1,\mathbf{1})+(\mathbf{1},+1,0,\mathbf{1})+\text{c.c}$	1	$\ell=3$	0	$(0,0,1)$
Θ_1	$(\mathbf{27},0,+\frac12,\mathbf{1})+(\mathbf{1},0,+\frac32,\mathbf{1})$	16	$\eta=(\frac12,\frac12,0)$	0	$(\frac12,\frac12,0)$
Θ_1	$(\mathbf{1},\pm\frac12,0,\mathbf{1})$	32	$\eta=(\frac12,\frac12,0)$	1	$(\frac12,\frac12,0)$
Θ_2	$(\mathbf{27},+\frac14,-\frac14,\mathbf{1})+(\mathbf{1},+\frac34,-\frac34,\mathbf{1})$	16	$\eta=(\frac12,0,\frac12)$	0	$(\frac12,0,\frac12)$
Θ_2	$(\mathbf{1},\pm\frac14,\pm\frac34,\mathbf{1})$	32	$\eta=(\frac12,0,\frac12)$	1	$(\frac12,0,\frac12)$
$\Theta_1\Theta_2$	$(\mathbf{27},-\frac14,-\frac14,\mathbf{1})+(\mathbf{1},-\frac34,-\frac34,\mathbf{1})$	16	$\eta=(0,\frac12,\frac12)$	0	$(0,\frac12,\frac12)$
$\Theta_1\Theta_2$	$(\mathbf{1},\mp\frac14,\pm\frac34,\mathbf{1})$	32	$\eta=(0,\frac12,\frac12)$	1	$(0,\frac12,\frac12)$

(the abelian gauge charges are normalized according to $k_{U(1)} = \left(\begin{smallmatrix}1&0\\0&3\end{smallmatrix}\right)$). Substituting the above spectrum and the values of q^i_I into eqs. (3.10), we arrive at

$$\alpha^i_{E(8)} \;=\; -30, \qquad \alpha^i_{E(6)} \;=\; +42, \qquad \alpha^i_{U(1)} \;=\; +42k_{U(1)}, \tag{C.13}$$

which clearly agrees with eqs. (C.11) and (3.22) are for $\delta^i_{\rm GS} = 0$.

Next consider the $Z_2 \times Z_2$ orbifold in which each of the three twists Θ_1, Θ_2 and $\Theta_1\Theta_2$ would break the $E_8 \times E_8$ down to a $G^{N=2} = E_7 \times SU(2) \times SO(16)$; the combined effect of these twists leaves the $N = 1$ orbifold with the unbroken gauge symmetry $G = E_6 \times U(1)^2 \times SO(8) \times SO(8)$. Naturally, for this orbifold, $\mathbf{b}^{N=2}_a(i)$ are given by eqs. (C.12), although a proper interpretation of that result requires paying careful attention to the way G is embedded into $G^{N=2}(i)$, which is different for different i. Thus,

$$\mathbf{b}^{N=2}_{E(6)}(i) \;=\; -12, \quad \mathbf{b}^{N=2}_{SO(8)}(i) \;=\; \mathbf{b}^{N=2}_{SO(8)'}(i) \;=\; +36, \tag{C.14}$$

$$\mathbf{b}^{N=2}_{U(1)}(1) \;=\; \begin{pmatrix} 36 & -144 \\ -144 & 396 \end{pmatrix}, \quad \mathbf{b}^{N=2}_{U(1)}(2) \;=\; \begin{pmatrix} 36 & +144 \\ +144 & 396 \end{pmatrix}, \quad \mathbf{b}^{N=2}_{U(1)}(3) \;=\; \begin{pmatrix} 180 & 0 \\ 0 & -36 \end{pmatrix},$$

where on the first line $i = 1, 2, 3$ and on the second line we use the same basis for the two abelian charges as in the previous $Z_2 \times Z_2$ example. The matter field spectrum of the present orbifold is

sector	$E_6 \times U(1)^2 \times SO(8) \times SO(8)$	#	ℓ_I or η^i_I	osc.	avg. q^i_I
untwisted	$(\mathbf{27},+\frac12,+\frac12,\mathbf{1},\mathbf{1})+(\mathbf{1},-\frac12,+\frac32,\mathbf{1},\mathbf{1})+\text{c.c}$	1	$\ell=1$	0	$(1,0,0)$
untwisted	$(\mathbf{1},0,0,\mathbf{8},\mathbf{8})$	1	$\ell=1$	0	$(1,0,0)$
untwisted	$(\mathbf{27},-\frac12,+\frac12,\mathbf{1},\mathbf{1})+(\mathbf{1},+\frac12,+\frac32,\mathbf{1},\mathbf{1})+\text{c.c}$	1	$\ell=2$	0	$(0,1,0)$
untwisted	$(\mathbf{1},0,0,\mathbf{8}',\mathbf{8}')$	1	$\ell=2$	0	$(0,1,0)$
untwisted	$(\mathbf{27},0,-1,\mathbf{1},\mathbf{1})+(\mathbf{1},+1,0,\mathbf{1},\mathbf{1})+\text{c.c}$	1	$\ell=3$	0	$(0,0,1)$
untwisted	$(\mathbf{1},0,0,\mathbf{8}'',\mathbf{8}'')$	1	$\ell=3$	0	$(0,0,1)$
Θ_1	$(\mathbf{1},+\frac12,0,\mathbf{1},\mathbf{8}'')+(\mathbf{1},-\frac12,0,\mathbf{8}'',\mathbf{1})$	16	$\eta=(\frac12,\frac12,0)$	0	$(\frac12,\frac12,0)$
Θ_2	$(\mathbf{1},-\frac14,-\frac34,\mathbf{1},\mathbf{8}')+(\mathbf{1},+\frac14,+\frac34,\mathbf{8}',\mathbf{1})$	16	$\eta=(\frac12,0,\frac12)$	0	$(\frac12,0,\frac12)$
$\Theta_1\Theta_2$	$(\mathbf{1},-\frac14,+\frac34,\mathbf{1},\mathbf{8})+(\mathbf{1},+\frac14,-\frac34,\mathbf{8},\mathbf{1})$	16	$\eta=(0,\frac12,\frac12)$	0	$(0,\frac12,\frac12)$

which leads to the following values of the modular anomalies (3.10):

$$\alpha^i_{E(6)} = -6, \qquad \alpha^i_{SO(8)} = \alpha^i_{SO(8)'} = +18, \tag{C.15}$$

$$\alpha^1_{U(1)} = \begin{pmatrix} 18 & -72 \\ -72 & 198 \end{pmatrix}, \qquad \alpha^2_{U(1)} = \begin{pmatrix} 18 & +72 \\ +72 & 198 \end{pmatrix}, \qquad \alpha^3_{U(1)} = \begin{pmatrix} 90 & 0 \\ 0 & -18 \end{pmatrix}.$$

Comparing these results with eqs. (C.14), we see that again eqs. (3.22) hold true for $\delta^i_{\rm GS} = 0$. In fact, we shall momentarily see that $\delta^i_{\rm GS} = 0$ for all five $Z^2 \times Z^2$ orbifolds.

For the next $Z_2 \times Z_2$ orbifold, the little groups D_i of the three internal planes act similarly on the first E_8 but not on the second E_8. This time, the $N = 1$ gauge group is $G = E_6 \times U(1)^2 \times SO(16)$ and the $N = 2$ renormalization group coefficients $\mathbf{b}^{N=2}_a(i)$ are given by eqs. (C.14) for $i = 1, 2$ but by eq. (C.11) for $i = 3$. The matter fields of this orbifold are

sector	$E_6 \times U(1)^2 \times SO(16)$	#	ℓ_I or η^i_I	osc.	avg. q^i_I
untwisted	$(\mathbf{27}, +\frac{1}{2}, +\frac{1}{2}, \mathbf{1}) + (\mathbf{1}, -\frac{1}{2}, +\frac{3}{2}, \mathbf{1})$ + c.c	1	$\ell = 1$	0	$(1, 0, 0)$
untwisted	$(\mathbf{27}, -\frac{1}{2}, +\frac{1}{2}, \mathbf{1}) + (\mathbf{1}, +\frac{1}{2}, +\frac{3}{2}, \mathbf{1})$ + c.c	1	$\ell = 2$	0	$(0, 1, 0)$
untwisted	$(\mathbf{27}, 0, -1, \mathbf{1}) + (\mathbf{1}, +1, 0, \mathbf{1})$ + c.c	1	$\ell = 3$	0	$(0, 0, 1)$
untwisted	$(\mathbf{1}, 0, 0, \mathbf{128})$	1	$\ell = 3$	0	$(0, 0, 1)$
Θ_1	$(\mathbf{27}, 0, +\frac{1}{2}, \mathbf{1}) + (\mathbf{1}, 0, +\frac{3}{2}, \mathbf{1})$	16	$\eta = (\frac{1}{2}, \frac{1}{2}, 0)$	0	$(\frac{1}{2}, \frac{1}{2}, 0)$
Θ_1	$(\mathbf{1}, \pm\frac{1}{2}, 0, \mathbf{1})$	32	$\eta = (\frac{1}{2}, \frac{1}{2}, 0)$	1	$(\frac{1}{2}, \frac{1}{2}, 0)$
Θ_2	$(\mathbf{1}, +\frac{1}{4}, +\frac{3}{4}, \mathbf{16})$	16	$\eta = (\frac{1}{2}, 0, \frac{1}{2})$	0	$(\frac{1}{2}, 0, \frac{1}{2})$
$\Theta_1\Theta_2$	$(\mathbf{1}, -\frac{1}{4}, +\frac{3}{4}, \mathbf{16})$	16	$\eta = (0, \frac{1}{2}, \frac{1}{2})$	0	$(0, \frac{1}{2}, \frac{1}{2})$

Substituting this spectrum into eqs. (3.10), we obtain

$$\alpha^{1,2}_{E(6)} = -6, \qquad \alpha^{1,2}_{SO(16)} = +18, \qquad \alpha^{1,2}_{U(1)} = \begin{pmatrix} 18 & \mp 72 \\ \mp 72 & 198 \end{pmatrix},$$

$$\alpha^3_{E(6)} = +42, \qquad \alpha^3_{SO(16)} = -30, \qquad \alpha^3_{U(1)} = \begin{pmatrix} 42 & 0 \\ 0 & 126 \end{pmatrix} = +42k_{U(1)}, \tag{C.16}$$

which indeed agrees with eqs. (3.22) for $\delta^1_{\rm GS} = \delta^2_{\rm GS} = \delta^3_{\rm GS} = 0$.

In the remaining two $Z_2 \times Z_2$ orbifolds, the little group D_3 of the third internal plane acts differently from D_1 and D_2 on both E_8 factors: In both models, D_1 and D_2 break the first E_8 down to an $E_7 \times SU(2)$ and the second E_8 down to an $SO(16)$ while D_3 breaks the second E_8 down to an $E_7 \times SU(2)$; the difference is whether D_3 leaves the first E_8 unbroken or breaks it down to an $SO(16)$. In the first case, the surviving gauge group is $G = E_7 \times SU(2) \times SU(8) \times U(1)$ and the spectrum of the matter fields consists of

sector	$E_7 \times SU(2) \times SU(8) \times U(1)$	#	ℓ_I or η^i_I	osc.	avg. q^i_I
untwisted	$(\mathbf{1}, \mathbf{1}, \mathbf{28}, -\frac{1}{2})$ + c.c	1	$\ell = 1$	0	$(1, 0, 0)$
untwisted	$(\mathbf{1}, \mathbf{1}, \mathbf{28}, +\frac{1}{2})$ + c.c	1	$\ell = 2$	0	$(0, 1, 0)$
untwisted	$(\mathbf{1}, \mathbf{1}, \mathbf{70}, 0) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \pm 1)$	1	$\ell = 3$	0	$(0, 0, 1)$
untwisted	$(\mathbf{56}, \mathbf{2}, \mathbf{1}, 0)$	1	$\ell = 3$	0	$(0, 0, 1)$
Θ_1	$(\mathbf{1}, \mathbf{1}, \mathbf{28}, 0)$	16	$\eta = (\frac{1}{2}, \frac{1}{2}, 0)$	0	$(\frac{1}{2}, \frac{1}{2}, 0)$
Θ_1	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \pm\frac{1}{2})$	32	$\eta = (\frac{1}{2}, \frac{1}{2}, 0)$	1	$(\frac{1}{2}, \frac{1}{2}, 0)$
Θ_2	$(\mathbf{1}, \mathbf{2}, \bar{\mathbf{8}}, -\frac{1}{4})$	16	$\eta = (\frac{1}{2}, 0, \frac{1}{2})$	0	$(\frac{1}{2}, 0, \frac{1}{2})$
$\Theta_1\Theta_2$	$(\mathbf{1}, \mathbf{2}, \bar{\mathbf{8}}, +\frac{1}{4})$	16	$\eta = (0, \frac{1}{2}, \frac{1}{2})$	0	$(0, \frac{1}{2}, \frac{1}{2})$

where the abelian gauge charge is normalized to $k_{U(1)} = 1$. Therefore,

$$\begin{aligned} \alpha^{1,2}_{SU(8)} &= \alpha^{1,2}_{U(1)} = +18, \quad \alpha^{1,2}_{E(7)} = -6, \quad \alpha^{1,2}_{SU(2)} = +90, \\ \alpha^3_{SU(8)} &= \alpha^3_{U(1)} = +42, \quad \alpha^3_{E(7)} = \alpha^3_{SU(2)} = -30, \end{aligned} \tag{C.17}$$

which agrees with the fact that for this model, $\mathbf{b}^{N=2}_a(i)$ are given by eqs. (C.12) for $i = 1, 2$ and by eqs. (C.11) for $i = 3$.

The unbroken gauge symmetry of the other model is $G = SO(12)\times SU(2)\times SU(2)\times SU(8)\times U(1)$ while its matter fields are as follows:

sector	$SO(12)\times SU(2)\times SU(2)\times SU(8)\times U(1)$	#	ℓ_I or η_I^i	osc.	avg. q_I^i
untwisted	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{28},-\frac{1}{2})+\text{c.c}$	1	$\ell=1$	0	$(1,0,0)$
untwisted	$(\mathbf{32},\mathbf{2},\mathbf{1},\mathbf{1},0)$	1	$\ell=1$	0	$(1,0,0)$
untwisted	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{28},+\frac{1}{2})+\text{c.c}$	1	$\ell=2$	0	$(0,1,0)$
untwisted	$(\mathbf{32'},\mathbf{1},\mathbf{2},\mathbf{1},0)$	1	$\ell=2$	0	$(0,1,0)$
untwisted	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{70},0)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\pm1)$	1	$\ell=3$	0	$(0,0,1)$
untwisted	$(\mathbf{12},\mathbf{2},\mathbf{2},\mathbf{1},0)$	1	$\ell=3$	0	$(0,0,1)$
Θ_1	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},-\frac{1}{2})$	16	$\eta=(\frac{1}{2},\frac{1}{2},0)$	0	$(\frac{1}{2},\frac{1}{2},0)$
Θ_1	$(\mathbf{12},\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2})$	16	$\eta=(\frac{1}{2},\frac{1}{2},0)$	0	$(\frac{1}{2},\frac{1}{2},0)$
Θ_2	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{8},+\frac{1}{4})$	16	$\eta=(\frac{1}{2},0,\frac{1}{2})$	0	$(\frac{1}{2},0,\frac{1}{2})$
$\Theta_1\Theta_2$	$(\mathbf{1},\mathbf{2},\mathbf{1},\bar{\mathbf{8}},+\frac{1}{4})$	16	$\eta=(0,\frac{1}{2},\frac{1}{2})$	0	$(0,\frac{1}{2},\frac{1}{2})$

Consequently,

$$\begin{gathered}\alpha^1_{SO(12)} = \alpha^1_{SU(2)} = \alpha^2_{SO(12)} = \alpha^2_{SU(2)'} = \alpha^3_{SU(8)} = -6,\\ \alpha^1_{SU(2)'} = \alpha^2_{SU(2)} = \alpha^3_{U(1)} = +90,\\ \alpha^1_{SU(8)} = \alpha^1_{U(1)} = \alpha^2_{SU(8)} = \alpha^2_{U(1)} = \alpha^3_{SU(8)} = \alpha^3_{SU(2)} = \alpha^3_{SU(2)'} = +18,\end{gathered} \tag{C.18}$$

which agrees with the fact that for this model, all the $\mathbf{b}_a^{N=2}(i)$ are given by eqs. (C.12), but the embedding of the $N=1$ gauge group into the gauge group of the $N=2$ D_i orbifold depends on i. Again, in both models, eqs. (3.22) hold for $\delta^i_{\text{GS}}=0$.

C.4. Other Orbifold Examples.

After the Z_3 and $Z_2\times Z_2$ orbifolds we have presented thus far, the next simplest group of orbifold examples consists of twelve Z_4 orbifolds whose rotation group is generated by $\Theta=(i,i,-1)$. (There are twelve inequivalent twists of the $E_8\times E_8$ Kac-Moody algebra that are compatible with this rotation.) For these orbifolds, the little groups of the first two planes are trivial while the little group of the third plane is a Z_2. Therefore, we expect

$$\alpha_a^{1,2} = k_a\,\delta^{1,2}_{\text{GS}} \qquad \alpha_a^3 = k_a\,\delta^3_{\text{GS}} + \tfrac{1}{2}\mathbf{b}_a^{N=2}, \tag{C.19}$$

where the $N=2$ beta-function coefficients $\mathbf{b}_a^{N=2}$ are given by eqs. (C.11) or (C.12), whichever is appropriate for a particular Z_4 orbifold. Without going through the spectra of the twelve orbifolds, let us simply state the results: For all twelve models, eqs. (C.19) are always satisfied and furthermore, $\delta^3_{\text{GS}}=0$. On the other hand, $\delta^{1,2}_{\text{GS}}$ depend on a particular model but generally do not vanish.

All of the above examples have a common feature that $\delta^i_{\text{GS}}=0$ whenever some twisted sectors leave the ith plane unrotated. However, a more general survey shows that δ^i_{GS} vanishes only when the little group D_i of the ith plane has index 2 (which happens to be the case for all the non-trivial little groups of the Z_3, $Z_2\times Z_2$ and Z_4 orbifolds). In particular, for the Z_6 orbifolds whose rotation group is generated by the $\Theta=(e^{2\pi i/6},e^{2\pi i/3},-1)$, the little groups are $D_1=1$, $D_2=Z_2$ and $D_3=Z_3$; consequently, $\delta^3_{\text{GS}}=0$ but $\delta^2_{\text{GS}}\neq 0$.

There are sixty one inequivalent Z_6 twists of the $E_8\times E_8$ Kac-Moody algebra that are compatible with $\Theta=(e^{2\pi i/6},e^{2\pi i/3},-1)$. The resulting list of sixty one models is clearly much too long to be presented here in full detail, so we decided to present only two of these Z_6 orbifolds as examples: The left-right symmetric $(2,2)$ orbifold whose gauge group is $\big(E_6\times U(1)^2\big)\times E_8$ and a $(0,2)$ Z_6 orbifold with $G=(SU(6)\times SU(3)\times SU(2))\times(SU(8)\times U(1))$; their respective spectra of the matter fields are listed in tables on the following pages. (The abelian gauge

Matter Fields of the (2,2) Z_6 Orbifold

sector	$E_6 \times U(1)^2 \times E_8$	#	ℓ_I or η_I^i	osc.	avg. q_I^i
untwisted	$(\mathbf{27}, +\frac{1}{2}, +\frac{1}{2}, \mathbf{1})$	1	$\ell = 1$	0	$(1,0,0)$
untwisted	$(\mathbf{1}, -1, 0, \mathbf{1}) + (\mathbf{1}, +\frac{1}{2}, -\frac{3}{2}, \mathbf{1})$	1	$\ell = 1$	0	$(1,0,0)$
untwisted	$(\mathbf{27}, -\frac{1}{2}, +\frac{1}{2}, \mathbf{1}) + (\mathbf{1}, -\frac{1}{2}, -\frac{3}{2}, \mathbf{1})$	1	$\ell = 2$	0	$(0,1,0)$
untwisted	$(\mathbf{27}, 0, -1, \mathbf{1}) + (\overline{\mathbf{27}}, 0, +1, \mathbf{1})$	1	$\ell = 3$	0	$(0,0,1)$
Θ	$(\mathbf{27}, -\frac{1}{12}, -\frac{1}{4}, \mathbf{1})$	12	$\eta = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$	0	$(\frac{5}{6}, \frac{2}{3}, \frac{1}{2})$
Θ	$(\mathbf{1}, -\frac{7}{12}, -\frac{3}{4}, \mathbf{1})$	12	$\eta = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$	1	$(\frac{11}{6}, \frac{2}{3}, \frac{1}{2})$
Θ	$(\mathbf{1}, +\frac{5}{12}, -\frac{3}{4}, \mathbf{1})$	24	$\eta = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$	1 or 2	$(\frac{11}{6}, \frac{7}{6}, \frac{1}{2})$
Θ	$(\mathbf{1}, -\frac{1}{12}, +\frac{3}{4}, \mathbf{1})$	48	$\eta = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$	1, 2 or 3	$(\frac{11}{6}, \frac{11}{12}, \frac{1}{2})$
Θ^2	$(\mathbf{27}, -\frac{1}{6}, +\frac{1}{2}, \mathbf{1}) + (\mathbf{1}, -\frac{1}{6}, -\frac{3}{2}, \mathbf{1})$	6	$\eta = (\frac{1}{3}, \frac{2}{3}, 0)$	0	$(\frac{2}{3}, \frac{1}{3}, 0)$
Θ^2	$(\overline{\mathbf{27}}, -\frac{1}{6}, -\frac{1}{2}, \mathbf{1}) + (\mathbf{1}, -\frac{1}{6}, +\frac{3}{2}, \mathbf{1})$	3	$\eta = (\frac{1}{3}, \frac{2}{3}, 0)$	0	$(\frac{2}{3}, \frac{1}{3}, 0)$
Θ^2	$(\mathbf{1}, -\frac{2}{3}, 0, \mathbf{1})$	9	$\eta = (\frac{1}{3}, \frac{2}{3}, 0)$	1	$(\frac{4}{3}, 0, 0)$
Θ^2	$(\mathbf{1}, +\frac{1}{3}, 0, \mathbf{1})$	24	$\eta = (\frac{1}{3}, \frac{2}{3}, 0)$	1 or 2	$(\frac{7}{6}, \frac{-1}{24}, 0)$
Θ^3	$(\mathbf{27}, +\frac{1}{4}, -\frac{1}{4}, \mathbf{1})$	8	$\eta = (\frac{1}{2}, 0, \frac{1}{2})$	0	$(\frac{1}{2}, 0, \frac{1}{2})$
Θ^3	$(\overline{\mathbf{27}}, -\frac{1}{4}, +\frac{1}{4}, \mathbf{1}) + (\mathbf{1}, \pm\frac{3}{4}, \mp\frac{3}{4}, \mathbf{1})$	4	$\eta = (\frac{1}{2}, 0, \frac{1}{2})$	0	$(\frac{1}{2}, 0, \frac{1}{2})$
Θ^3	$(\mathbf{1}, +\frac{1}{4}, +\frac{3}{4}, \mathbf{1})$	24	$\eta = (\frac{1}{2}, 0, \frac{1}{2})$	1	$(\frac{1}{2}, 0, \frac{1}{2})$
Θ^3	$(\mathbf{1}, -\frac{1}{4}, -\frac{3}{4}, \mathbf{1})$	20	$\eta = (\frac{1}{2}, 0, \frac{1}{2})$	1	$(\frac{7}{10}, 0, \frac{1}{2})$
Θ^4	$(\mathbf{27}, +\frac{1}{6}, +\frac{1}{2}, \mathbf{1}) + (\mathbf{1}, +\frac{1}{6}, -\frac{3}{2}, \mathbf{1})$	6	$\eta = (\frac{2}{3}, \frac{1}{3}, 0)$	0	$(\frac{1}{3}, \frac{2}{3}, 0)$
Θ^4	$(\overline{\mathbf{27}}, +\frac{1}{6}, -\frac{1}{2}, \mathbf{1}) + (\mathbf{1}, +\frac{1}{6}, +\frac{3}{2}, \mathbf{1})$	3	$\eta = (\frac{2}{3}, \frac{1}{3}, 0)$	0	$(\frac{1}{3}, \frac{2}{3}, 0)$
Θ^4	$(\mathbf{1}, +\frac{2}{3}, 0, \mathbf{1})$	9	$\eta = (\frac{2}{3}, \frac{1}{3}, 0)$	1	$(0, \frac{4}{3}, 0)$
Θ^4	$(\mathbf{1}, -\frac{1}{3}, 0, \mathbf{1})$	21	$\eta = (\frac{2}{3}, \frac{1}{3}, 0)$	1 or 2	$(\frac{1}{21}, \frac{23}{21}, 0)$

charges are normalized to $k_{U(1)} = \left(\begin{smallmatrix}1&0\\0&3\end{smallmatrix}\right)$ for the $(2,2)$ orbifold and $K_{U(1)} = 1$ for the $(0,2)$ orbifold.)

For the left-right symmetric Z_6 orbifold, the $N=2$ orbifolds produced by the little groups $D_2 = Z_2$ and $D_3 = Z_3$ are both left-right symmetric, and for all such orbifolds,

$$\mathbf{b}^{N=2}_{E(8)} \;=\; -60, \qquad \mathbf{b}^{N=2}_{E(6)} \;=\; +84, \qquad \mathbf{b}^{N=2}_{U(1)} \;=\; +84 k_{U(1)}\,. \qquad \text{(C.20)}$$

Matter Fields of a (0,2) Z_6 Orbifold

sector	$SU(6) \times SU(3) \times SU(2) \times SU(8) \times U(1)$	#	ℓ_I or η_I^i	osc.	avg. q_I^i
untwisted	$(\mathbf{6}, \mathbf{3}, \mathbf{2}, \mathbf{1}, 0)$	1	$\ell = 1$	0	$(1,0,0)$
untwisted	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{28}, +\frac{1}{2}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, -1)$	1	$\ell = 1$	0	$(1,0,0)$
untwisted	$(\mathbf{15}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, 0) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{28}, -\frac{1}{2})$	1	$\ell = 2$	0	$(0,1,0)$
untwisted	$(\mathbf{20}, \mathbf{1}, \mathbf{2}, \mathbf{1}, 0) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{70}, 0)$	1	$\ell = 3$	0	$(0,0,1)$
Θ^2	$(\overline{\mathbf{15}}, \mathbf{1}, \mathbf{1}, \mathbf{1}, +\frac{1}{3})$	6	$\eta = (\frac{1}{3}, \frac{2}{3}, 0)$	0	$(\frac{2}{3}, \frac{1}{3}, 0)$
Θ^2	$(\mathbf{6}, \mathbf{1}, \mathbf{2}, \mathbf{1}, +\frac{1}{3}) + (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, -\frac{2}{3})$	3	$\eta = (\frac{1}{3}, \frac{2}{3}, 0)$	0	$(\frac{2}{3}, \frac{1}{3}, 0)$
Θ^2	$(\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, +\frac{1}{3})$	9	$\eta = (\frac{1}{3}, \frac{2}{3}, 0)$	1	$(1, -\frac{1}{3}, 0)$
Θ^3	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \overline{\mathbf{8}}, +\frac{1}{4})$	8	$\eta = (\frac{1}{2}, 0, \frac{1}{2})$	0	$(\frac{1}{2}, 0, \frac{1}{2})$
Θ^3	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{8}, \frac{1}{4})$	4	$\eta = (\frac{1}{2}, 0, \frac{1}{2})$	0	$(\frac{1}{2}, 0, \frac{1}{2})$
Θ^4	$(\mathbf{15}, \mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{3})$	3	$\eta = (\frac{2}{3}, \frac{1}{3}, 0)$	0	$(\frac{1}{3}, \frac{2}{3}, 0)$
Θ^4	$(\overline{\mathbf{6}}, \mathbf{1}, \mathbf{2}, \mathbf{1}, -\frac{1}{3}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, +\frac{2}{3})$	6	$\eta = (\frac{2}{3}, \frac{1}{3}, 0)$	0	$(\frac{1}{3}, \frac{2}{3}, 0)$
Θ^4	$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, -\frac{1}{3})$	9	$\eta = (\frac{1}{3}, \frac{2}{3}, 0)$	1	$(-\frac{1}{3}, 1, 0)$

At the same time, eqs. (3.10) give us

$$\begin{aligned}
\alpha^1_{E(8)} &= -30\,, \quad \alpha^1_{E(6)} = -30\,, \quad \alpha^1_{U(1)} = \begin{pmatrix} -30 & 0 \\ 0 & -90 \end{pmatrix} = -30 k_{U(1)}\,, \\
\alpha^2_{E(8)} &= -30\,, \quad \alpha^2_{E(6)} = +18\,, \quad \alpha^2_{U(1)} = \begin{pmatrix} +18 & 0 \\ 0 & +54 \end{pmatrix} = +18 k_{U(1)}\,, \qquad \text{(C.21)} \\
\alpha^3_{E(8)} &= -30\,, \quad \alpha^3_{E(6)} = +42\,, \quad \alpha^3_{U(1)} = \begin{pmatrix} +42 & 0 \\ 0 & +126 \end{pmatrix} = +42 k_{U(1)}\,,
\end{aligned}$$

and we immediately see that eqs. (3.22) are satisfied for

$$\delta^1_{\rm GS} \;=\; -30, \qquad \delta^2_{\rm GS} \;=\; -10, \qquad \delta^3_{\rm GS} \;=\; 0. \qquad \text{(C.22)}$$

For the other Z_6 example, the $N=2$ orbifold produced by the little group of the second plane has $G^{N=2} = E_7 \times SU(2) \times SO(16)$ and its beta-function coefficients are given by eq. (C.12); similarly, for the little group of the third plane, $G^{N=2} = E_6 \times SU(3) \times E_7 \times U(1)$ and the beta-function coefficients are

$$\mathbf{b}^{N=2}_{E(6)} \;=\; \mathbf{b}^{N=2}_{SU(3)} \;=\; +48, \qquad \mathbf{b}^{N=2}_{E(7)} \;=\; -24, \qquad \mathbf{b}^{N=2}_{U(1)} \;=\; +120. \qquad \text{(C.23)}$$

At the same time, the modular anomaly coefficients (3.10) of this Z_6 orbifold are

$$\begin{aligned}
\alpha^1_{SU(6)} \;=\; \alpha^1_{SU(3)} \;=\; \alpha^1_{SU(2)} \;=\; \alpha^1_{SU(8)} \;=\; \alpha^1_{U(1)} \;=\; +2, \qquad\qquad \text{(C.24)}\\
\alpha^2_{SU(6)} \;=\; \alpha^2_{SU(3)} \;=\; -2, \qquad \alpha^2_{SU(2)} \;=\; +62, \qquad \alpha^2_{SU(8)} \;=\; \alpha^2_{U(1)} \;=\; +14,\\
\alpha^3_{SU(6)} \;=\; \alpha^3_{SU(3)} \;=\; \alpha^3_{SU(2)} \;=\; +24, \qquad \alpha^3_{SU(8)} \;=\; -12, \qquad \alpha^3_{U(1)} \;=\; +60,
\end{aligned}$$

which satisfies eqs. (3.22) for

$$\delta^1_{\rm GS} \;=\; \delta^2_{\rm GS} \;=\; +2, \qquad \delta^3_{\rm GS} \;=\; 0. \tag{C.25}$$

Notice that for both examples, $\delta^2_{\rm GS} \neq 0$ even though the little group of the second plane is non-trivial; on the other hand, $\delta^3_{\rm GS} = 0$. From the orbifolds we have studied so far, it appears that $\delta^i_{\rm GS}$ vanishes whenever the little group of the ith plane has index two (*e.g.*, $Z_2 \subset Z_2^2$, $Z_2 \subset Z_4$ or $Z_3 \subset Z_6$). For the orbifold in which the second E_8 remains unbroken, this behavior results from the absence of any E_8-charged massless matter fields (which implies $\alpha^i_{E(8)} = -T(E_8) = \frac{1}{2}\mathbf{b}^{N=2}_{E(8)}$), but we have no idea why the $(0,2)$ orbifolds in which both E_8's are broken also follow the same pattern.

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